УДК 1(091) + 16 + 17 + 51-7 + 510.687 + 510.82 + 512

HILBERT'S CONCEPTION OF "EXISTENCE-IN-MATHEMATICS", AND MODELLING IT BY A FORMAL AXIOMATIC THEORY Ф+∃ TREATING EXISTENCE NOT AS THE QUANTIFIER BUT AS A MODALITY

V. O. Lobovikov
Institute of Philosophy and Law UB RAS (Yekaterinburg)
vlobovikov@mail.ru

Abstract. The *subject-matter* of investigation is Hilbert's-principle consisting of two parts: (a) proclaiming equivalence-of-consistency-and-truth-in-mathematics; (b) proclaiming equivalence-of-consistency-and-existence-in-mathematics. The *target* – explication and vindication of the principle. The *scientific novelty*: for reaching the goal, (1) a hitherto unknown logically formalized multimodal axiomatic epistemology-and-ontology-system called Ф+∃ has been constructed; (2) by means of Ф+∃, a precise *axiomatic* definition of the notion "existence as modality" is submitted for the first time; (3) by means of artificial language of Ф+∃, a precise formulation of the principle-of-equivalence-of-consistency-and-existence is given; (4) for the first time, the article presents *formal deductive inferences* (in formal-theory-Ф+∃) of such formulae which make up Hilbert's principle (given an appropriate interpretation of these formulae).

Keywords: existence-in-mathematics, Hilbert, principle-of-equivalence-of-consistency-and-existence, principle-of-equivalence-of-consistency-and-truth, formal-multimodal-axiomatic-theory-Ф+∃.

For citation: Lobovikov, V. O. (2024). Hilbert's Conception of "Existence-in-Mathematics" and Modelling it by a Formal Axiomatic Theory Ф+∃ Treating Existence not the Quanifier but as a Modality. *Respublica Literaria*. Vol. 5. No. 1. Pp. 16-50. DOI: 10.47850/RL.2024.5.1.16-50

ГИЛЬБЕРТОВСКАЯ КОНЦЕПЦИЯ «СУЩЕСТВОВАНИЯ В МАТЕМАТИКЕ» И ЕЕ МОДЕЛИРОВАНИЕ ФОРМАЛЬНОЙ АКСИОМАТИЧЕСКОЙ ТЕОРИЕЙ Ф+∃, РАССМАТРИВАЮЩЕЙ СУЩЕСТВОВАНИЕ НЕ КАК КВАНТОР, А КАК МОДАЛЬНОСТЬ

В. О. Лобовиков
Институт философии и права Уральского отделения РАН (г. Екатеринбург)
vlobovikov@mail.ru

Аннотация. *Предмет* исследования – принцип Гильберта, состоящий из двух частей: (a) утверждение об эквивалентности непротиворечивости и истины в математике; (b) утверждение об эквивалентности непротиворечивости и существования в математике. *Цель* – уточнение и обоснование упомянутого принципа. *Научная новизна*: *для* достижения этой цели, (1) построена некая пока неизвестная логически формализованная мультимодальная аксиоматическая система онтологии и эпистемологии, названная Ф+∃; (2) впервые с помощью Ф+∃ дано точное *аксиоматическое* определение понятия «существование как модальность»; (3) на искусственном языке Ф+∃ точно сформулирован принцип эквивалентности непротиворечивости и существования в математике; (4) впервые публикуются *формальные дедуктивные выводы* (в формальной теории Ф+∃) тех формул, которые вместе образуют (в соответствующей интерпретации) принцип Гильберта.

Ключевые слова: существование в математике, Гильберт, принцип эквивалентности существования и непротиворечивости, принцип эквивалентности истины и непротиворечивости, формальная мультимодальная аксиоматическая теория Ф+∃.

Для цитирования: Лобовиков, В. О. (2024). Гильбертовская концепция «существования в математике» и ее моделирование формальной аксиоматической теорией Ф+∃, рассматривающей существование не как квантор, а как модальность. *Respublica Literaria*. Т, 5. № 1. С. 16-50. DOI: 10.47850/RL.2024.5.1.16-50

1. Introduction

"Hilbert's Principle: in mathematics, if it is consistent for something to exist then it does exist" [Doherty, 2017, p. 107].

***

"What does the word exist mean in mathematics? It means, I said, to be free from contradiction" [Poincaré, 1912, p. 526]

In philosophical logic and metaphysics, criterions of existence and of truth make up one of the very old and hard problems since the ancient Greek philosophy times to our days [Plato, 1994; Aristotle, 1994; Юм, 1965; Leibniz, 1969, 1981; Kant, 1994, 1996; Frege, 1971, 1980, 1984; Hilbert, 1990, 1996a, 1996b, 1996c; et al.], are implied here. In modern philosophy, there had been a *debate of existence as a predicate* [Kant, 1994; Frege, 1971, 1980, 1984; Hintikka, 1986; Berto, 2012; Haaparanta, 1985, 1986; Ершов, Самохвалов, 2007; Целищев, 2003; Nelson, 2022]. While discussing the ontological proofs of God's existence, Immanuel Kant argued that *existence was not a real predicate* [Kant, 1994]. Gottlob Frege significantly *explicated* this thesis (he made it formulated more precisely) by means of mathematizing logic systematically and claiming that *existence is a second-order predicate*. However, the tendency (initiated by Alexius Meinong [Berto and Priest, 2014; Fine, 1984, 1985; Jacquette, 1996, 1997, 2015; Marek, 2022; Meinong, 1960; Parsons, 1980, 1982; Hintikka, 1984; Zalta, 1983]) somehow to explicate and accept (the well-explicated option of) the extraordinary Meinongian idea of existence of nonexistent objects makes the contemporary situation not so easy and clear as it has been seemed to followers of Kant and Frege. In this connection, I agree with Maria Reicher when she writes: "It turns out that Kant's view that "exists" is not a "real" predicate and Frege's view, that *"exists" is not a predicate of individuals* (i.e., a predicate that yields a well-formed sentence if one puts a singular term in front of it), has to be abandoned if one is to accept the claim that there are nonexistent objects" [Reicher, 2022]. An intellectually respectable critique of the philosophical debates of *existence as a predicate*, and a reasonable suggestion of solving the problem is presented in [Ершов, Самохвалов, 2007, с. 41-62, 118-139; Целищев, 2003, с. 20-22].

However, in spite of the fact that among philosophers the discussion still goes on, in the present article, I shall not go into details of the debate (of existence as a predicate) and shall not take part in it. Also, I shall not discuss the well-elaborated mathematical logic doctrine of *existence as the quantifier*. In XX century, B. Russel, A. Prior, W. Quine, N. Resher, J. Hintikka, V. Tselishchev, and many other celebrated logicians had undertaken an all-around investigation of the philosophical ontology problems raising in relation to the very influential tendency to exploit various mathematical logic theories of quantification in general and of existence quantifier especially for clarifying the proper philosophical (ontological) notion "existence (in general)". An in-depth philosophical ontology discussion and logic analysis of the theme "Existence and Quantification" may be found, for instance, in [Resher, 1959; Prior, 1961; Hintikka, 1969; Quine, 1969, 1980; Целищев, 2010].

As, today, existence as the quantifier is already well-studied, I intend, in this article, to concentrate on recognizing and investigating *existence as a modality*. Even at the very beginning of philosophy development it was recognized clearly that the notion "existence" is essential for metaphysics [Aristotle, 1994; Plato, 1994; Nelson, 2022]. Consequently, if one seriously considers *modal logic as metaphysics* (for example, [Williamson, 2013]), then it is highly likely that the one must recognize clearly and investigate systematically the nontrivial hypothesis that *existence is a modality*. In modern philosophical literature of formal ontology and logic of existence [Bader, 2021; Blecher, 2012; Rosefeldt, 2020], this hypothesis has been already discussed somehow (especially in writings on history of philosophy). For example, T Rosefeldt has argued that, for Kant, *existence is* (not a quantity, but) *a modality* [2020]. Moreover, in the internet, one can find even such an online publication, which is called "Existence as a modality" [Bader, 2021]. Also, one can find the title "Introduction: Existence as a Modality" in the PhD dissertation [Blecher, 2012].

In the present article I am to study the hypothesis (that *existence is a modality*) with a view for applying results of this study to explicating and vindicating Hilbert's principle in question. Thus, in this paper, I am to depart from *pure* philosophy (proper metaphysics) to *applied* one. I mean *applying* metaphysics (understood as *modal* logic) to philosophical grounding proper mathematics. Quite precise definitions of the proper philosophical notions "truth" and "existence" are very important (even indispensable) for sufficient clarifying the philosophical (especially logical) foundations of mathematics. Many celebrated mathematicians and logicians had been involved in creating (inventing) such definitions and systematical investigating them, for example, G. Frege, B. Russel, H. Poincaré, D. Hilbert, K. Gödel, H. Weyl, L. Brouwer, V. Glivenko, A. Heyting, S. Leśniewski; A. Tarski, A. Church, S. Kleeny, A. Kolmogorov, A. Markov Jr., Yu. Ershov, K. Samokhvalov, Ch. Chihara, et al. An intellectually respectable survey of the relevant scientific literature on the theme can be found in [Chihara, 1990; Mancosu, 1998; Moschovakis, 2007; Целищев, 2010, 2003; Zach, 2023]. Concerning David Hilbert's principles of (a) equivalence-of-consistency-and-truth and (b) equivalence-of-consistency-and-existence, it is worth taking into an account the following citation from his letter to Gottlob Frege:

> You write "From the truth of the axioms it follows that they do not contradict one another". It interested me greatly to read this sentence of yours, because in fact for as long as I have been thinking, writing and lecturing about such things, I have always said the very opposite: if arbitrarily chosen axioms together with everything which follows from them do not contradict one another, then they are true, and the things defined by the axioms exist. For me that is the criterion of truth and existence [Frege, 1980, pp. 39-40].

In relation to this extract from D. Hilbert's letter to G. Frege, F. T. Doherty writes in her interesting article:

> The … extract from Hilbert's letter has received much attention. On the basis of it, a general principle for mathematical ontology has been attributed to Hilbert which I call *Hilbert's Principle*:
>
> **Hilbert's Principle:** In mathematics, if it is consistent for something to exist then it does exist.
>
> It is important to note that Hilbert's Principle is not a summary of the quote from Hilbert's letter. It is the attempt to extract a *thesis* from Hilbert on the basis of his remark. A lesser-known proponent of the same view is Poincaré … [Doherty, 2017, pp. 107-108].

Following F. T. Doherty, hereafter in this paper I am to use the name "Hilbert's principle" systematically as well. Obviously, in some concrete relations, generally speaking, the principle by Hilbert is very strange; it is somewhat strange even with respect to mathematics as such. In some concrete relations, I agree with F. T. Doherty, when she writes the following:

> As a general approach to ontology, such a principle is unintuitive and highly unparsimonious. Even taking into account the restriction to mathematics, the view is controversial. Consistency is very plausibly a necessary condition for the existence of mathematical entities, but why should it be considered a *sufficient* one? To answer such a question, we must be careful to understand the context in which Hilbert's Principle is given and not to assess it in a philosophical vacuum. This would be unproductive because Hilbert's Principle is not, by itself, a fully-fleshed out thesis. For example, it tells us nothing of what is meant by consistency, or by what means consistency is to be secured, or what kinds of things are established to exist. Because of this, no proper assessment of Hilbert's Principle can be reached before reconstructing Hilbert's actual contention. Thus, the concern of this paper will be neither to attack *nor* to defend Hilbert's view, but to discover it. As such, the guiding question of the paper will be as follows:
>
> Qu. What does Hilbert mean by Hilbert's Principle?
>
> To answer this question, I propose that we begin with what is commonly regarded as a bad answer. Namely, that Hilbert's Principle is an anticipation of the completeness theorem. I will henceforth call this the misguided reading of Hilbert's Principle. In what follows, we will give ourselves the task of asking whether there is any truth to this bad answer, and of articulating precisely what is misguided about it. This will require attention to many considerations which will very nicely pave the way for us to develop an alternative, historically informed, good answer to (Qu) [Doherty, 2017, p. 108].

Actually, Hilbert had formulated the principle not in a philosophical vacuum; he had accepted and followed some important aspects of G. W. Leibniz's [1969, 1981] and I. Kant's [1994, 1996] philosophies of mathematics [Murawski, 2002; Lutskanov, 2010, p. 122; Shanker, 1988, pp. 246-247]. According to Kant's nontrivial philosophy, knowledge of any statement of proper mathematics is *a-priori* knowledge [Kant, 1994, 1996]. Here, also it is worth taking into an account that, according to St. Shanker, as a matter of fact, D. Hilbert "selected as one of the two theses which doctoral candidates were required to defend in public the proposition that the objections to Kant's theory of the a priori nature of arithmetical judgments are unfounded" [Shanker, 1988, pp. 246-247].

Therefore, I guess, that, being a "fully-fleshed out thesis", Hilbert's principle should be formulated as the following pair of conditional statements S1 and S2.

S1. If Kant's philosophy-statement "mathematics is *a priori* knowledge" is true, then, in mathematics, consistency is equivalent to existence.

S2. If Kant's philosophy-statement "mathematics is *a priori* knowledge" is true, then, in mathematics, consistency is equivalent to truth.

As following Kant's philosophy of mathematics, Hilbert had presumed that the antecedents of the two conditionals are obviously true [Lutskanov, 2010, p. 122], [Shanker, 1988, pp. 246-247], he had detached his principle (under the discussion) from the couple of conditionals by *modus ponens*.

In the present article I am to investigate such a nontrivial hypothesis according to which Hilbert's "fully-fleshed-out" principle, namely, conjunction (S1 & S2) is quite true. Thus, I am to vindicate (explain and defend) Hilbert's principle (explicated and interpreted as (S1 & S2)) by means of *formal logic deriving* it (namely, *formal inferring* (S1 & S2)) in a newly constructed (invented) *formal axiomatic theory* Ф+∃ hitherto never published elsewhere.

It is worth taking into an account here that, according to S2, not only existence, but also truth is necessarily involved into the formulation of Hilbert's principle. Therefore, explicating and vindicating Hilbert's principle requires treating not only *existence as a modality*, but also *truth as a modality*. According to the interesting paper "Truth as Modality" [Wolenski, 2016] and paper "Truth-Logics" [Wright, 1996, p. 71], such an attitude to truth is quite realizable. For more detailed information about the controversy between Frege and Hilbert, readers are advised to make acquaintance with the well-written articles on the theme [Blanchette, 1996, 2007, 2018].

With respect to progressive development of universal philosophical epistemology, I would like to attract special attention of readers to the important possibility significantly *to generalize* the conjunction (S1 & S2), which represents Hilbert's "fully-fleshed-out" principle of philosophy (ontology) of mathematics. I think that abstractly speaking in general, it is quite natural to move from the conjunction (S1 & S2) to the conjunction ($S1^*$ & $S2^*$) of the immediately following pair of statements.

$S1^*$. In any knowledge sphere, if knowledge is *a priori*, then consistency is equivalent to existence.

$S2^*$. In any knowledge sphere, if knowledge is *a priori*, then consistency is equivalent to truth.

The sentences $S1^*$ and $S2^*$ are substantial *generalizations* of S1 and S2, respectively. The conjunction (S1 & S2) represents Hilbert's "fully-fleshed-out" principle of philosophy (epistemology-and-ontology) of proper mathematics. The conjunction ($S1^*$ & $S2^*$) represents a more fundamental (universal) principle of proper philosophical epistemology-and-ontology.

Universal philosophical epistemology is a proper philosophical theory of *knowledge in general*. Hence, it is to take into an account and to combine consistently both special types of knowledge, namely, *a-priori* and empirical (*a-posteriori*) ones. Also, along with synthesizing the two kinds of knowledge, it is indispensable for the actually *universal* philosophical epistemology to study a *universal* for the two. (Therefore, the significantly new formal axiomatic philosophy theory Ф+∃ formulated below submits an exact representation of *the universal*.) Certainly, it is necessary for the actually *universal* philosophical epistemology to deal systematically with the *empirical* material (curious *facts* and noteworthy trends of knowledge revision and evolution) systematically studied by the *evolutionary epistemology* [Bradie and Harms, 2020] along with the *history* and methodology *of science*. Initially, the epistemic modal logic – mathematized logic of knowledge had been constructed as a special kind of *normal modal logic* (in that meaning of the term, which had been defined precisely by S. Kripke). Noteworthy discussions of this kind (and stage of evolution) of epistemic modal logic may be found in [Wright, 1951; Hintikka, 1962, 1974; Rendsvig, Symons, Wang, 2023]. Since its creation to our days, the normal modal logic of knowledge has generated many grave paradoxes showing convincingly that the *normal* epistemic modal logic is to be rejected as subjectively counterintuitive and objectively inadequate. I believe that this is so because, for instance, the theorem of "*normal*" epistemic-modal-logic "If a person knows that p, then p" is in antagonistic contradiction with the well-known *empirical* theory of *evolution of knowledge* implying the intellectually respectable conception of *knowledge revision*. I believe that nowadays there is a strong want to dissolve the mentioned grave paradoxes of *normal*-epistemic-modal-logic by creating (inventing) a qualitatively new multimodal axiomatic formal-philosophy system combining actually universal epistemology with formal axiology and proper philosophical ontology. I guess that it is highly likely that such a novel multimodal axiomatic system of metaphysics would be *not a normal modal logic* system (in that meaning of the term which has been defined by S. Kripke). I guess that within the hypothetical new logically formalized multimodal axiomatic system of epistemology-and-ontology, one can construct a *formal inference* of a formula representing Hilbert's principle in this formalized system. In the present paper, the guess of mine is to be examined by (1) constructing *the formal theory* and (2) by constructing *the formal inference* of the formula representing Hilbert's principle in that formal theory.

2. Methods and Systems to be Applied to the Subject-Matter (Precise Definitions of Basic Notions Necessarily Used for Obtaining Novel Scientific Results)

Below in 2.1 and 2.2, such a substantial portion of my own previously published text is placed which has been already used by me many times, for example, in [Lobovikov, 2020, 2021, 2022, 2023; Лобовиков, 2023а, 2023б]. However, I firmly believe that this is not a scientist-misconduct (author delinquency) labeled "self-plagiarism (understood as *redundant* self-citations combined with *redundant* self-references)". The firm belief of mine is based upon the fact that the below-located (in sections 2.1 and 2.2) repetition of some significant aspects of my already published texts is *absolutely necessary* for blind-reviewer's and reader's adequate understanding and autonomous rechecking the *substantially new* scientific results *hitherto never published* elsewhere. If the mentioned significant quantity of self-citations and self-references is resolutely cut off (by editor's "Occam razor") and excluded from this article, then the article would become absolutely incomprehensible and unverifiable for the blind-reviewers and readers, consequently, the article would become *not proper scientific* one (according to K. Popper's principle of *falsifiability* of proper scientific statements). Thus, the self-citations and self-references (located in sections 2.1 and 2.2 of the present article) are vindicated by their *emergent necessity* for the *proper scientific* communication. Here I imply the emergent necessity perfectly to introduce the *unhabitual* methods and formal theories (along with the *not-well-known* basic notions) to be utilized indispensably for obtaining novel nontrivial scientific results unpublished hitherto. Evidently, using the word-combination "formal philosophy" in this paper is not a scientific novelty as the word-combination is already used systematically in modern scientific literature, for example, in [Montague, 1960; Thomason, 1974]. However, in the present article, the word-combination "formal philosophy" is used in a substantially different meaning to be clarified by the below-following text (although, certainly, the meanings under comparison are somehow connected in some nontrivial sense).

2.1. A New Formal Multimodal Axiomatic Theory Ф+∃ (Syntax Aspect)

The multimodal formal axiomatic system Ф+∃ is an outcome of significant enrichment and generalization of the formal axiomatic *epistemology-and-axiology* theories Σ [Lobovikov, 2020], Σ+C [Lobovikov, 2021], Σ+2C [Lobovikov, 2022, 2023], and Ф [Лобовиков, 2023а, 2023б]. In the formal theory Ф+∃ (which substantially generalizes the formal systems Σ, Σ+C, Σ-2C, and Ф), proper philosophical *ontology* is united naturally with universal philosophical *epistemology* and with formal *axiology*. In the present article, the result of such synthesizing is applied to philosophical foundations of mathematics in general, and to Hilbert's principle under investigation especially.

For precise defining the formal axiomatic theory Ф+∃, it is indispensable to begin with precise defining the concepts: "*alphabet* of object-language of Ф+∃"; "*term* of Ф+∃"; "*formula* of Ф+∃"; "*axiom* of Ф+∃". Precise definitions of these concepts of Ф+∃ *look similar* to the precise definitions of the corresponding concepts of Σ, Σ+C, Σ-2C, and Ф, which definitions are already published, respectively, in [Lobovikov, 2020, 2021, 2022, 2023; Лобовиков, 2023а, 2023б]. That is why one can have a strong impression of analogousness (déjà vu) and the *illusion* of identity. However, strictly speaking, in this paper, it is indispensable manifestly to give exact definitions of "*alphabet* of object-language of Ф+∃", "*term* of Ф+∃", "*formula* of Ф+∃", and "*axiom* of Ф+∃", notwithstanding the indicated similarity, as the notions "*similarity" and "identity" are not logically equivalent ones*; the relevant concepts of Σ, "Σ+C", "Σ+2C", and Ф *differ significantly* from the corresponding *similar* concepts of Ф+∃. Therefore, let us begin accurate formulating the definitions necessary for adequate understanding the article, notwithstanding the false impression that they are mere repetitions of the statements which are already published. Let us start with exact defining the concept "alphabet-of-object-language of formal-theory Ф+∃".

According to the below-given definition (consisting of 11 items), the alphabet-of-object-language of formal-theory Φ+∃ contains all the symbols which belong to the alphabets of object-languages of formal theories Σ, Σ+C, Σ-2C, and Φ. But, generally speaking, the conversion of this statement is not true, because, in Φ+∃, some very important novel sign is added to the alphabets of object-languages of Σ, Σ+C, Σ-2C, and Φ. The result of such a significant change (complementation) is the below-located precise definition of the alphabet-of-object-language of Φ+∃.

1. The lowercase Latin letters p, q, d (and the same letters possessing lower number indexes) are elements of the alphabet-of-object-language of Φ+∃. Such and only such lowercase Latin letters are called "*dictum variables*". In the alphabet of object-language of Φ+∃, *not all lowercase Latin letters are called dictum variables* because, according to the provided definition, such lowercase Latin letters which are elements of the set {g, b, e, n, x, y, z, a, s, h, t, f} do not belong to the set of *dictum variables* of object-language of Φ+∃.

2. The lowercase Latin letters a, s, h (and the same letters possessing lower literal indexes: a_t, s_m, h_s) are elements of the alphabet of object-language of Φ+∃. Such and only such lowercase Latin letters are called "*dictum constants*".

3. The well-known proper (pure) logic symbols ¬, ⊃, ↔, &, ∨ called, respectively, "classical negation", "classical (or 'material') implication", "classical equivalence", "classical conjunction", "classical not-excluding disjunction" are elements of the alphabet of object-language of Φ+∃.

4. Elements of the set {□, Ж, K, A, E, S, G, W, O, B, C, Y, T, F, P, D, U, J}, containing the sign □, the capital Cyrillic letter Ж, and some (but not all) capital Latin letters possessing no indexes, belong to the alphabet of object-language of Φ+∃. Such elements of the alphabet are called "modality symbols" in Φ+∃. The modality symbol Ж belongs *exclusively* to the alphabet of object-language of Φ+∃. The hitherto investigated formal philosophy systems Σ, Σ+C, Σ-2C, and Φ do not have the sign Ж in the alphabets of their object-languages.

5. The lowercase Latin letters *x*, *y*, *z* (and the same letters possessing lower number indexes) are elements of the alphabet-of-object-language of Φ+∃. Such and only such letters are called "*axiological variables*" in Φ+∃.

6. The lowercase Latin letters "g" and "b" called "*axiological constants*" also belong to the alphabet-of-object-language of Φ+∃.

7. The capital Latin letters possessing number indexes – E^1, C^1, K^1, K^2, E^2, C^2, C_j^n, B_i^n, D_m^n, A_k^n, … belong to the alphabet-of-object-language of Φ+∃. Such capital Latin letters are called "*axiological-value-functional symbols*". In these symbols, the upper number index *n* implies that the axiological-value-functional symbol (indexed by *n*) is *n*-placed one. Some axiological-value-functional symbols may have no lower number index. However, when value-functional symbols have lower number indexes, then, when these indexes are different, then the indexed value-functional symbols are different ones.

8. The signs "(" and ")" called "round brackets" belong to the alphabet of object-language of Φ+∃. These auxiliary signs are exploited in the given paper as usually in mathematical logic, namely, as pure technical symbols.

9. The signs "[" and "]" (called "square brackets") belong to the alphabet of object-language of Ф+∃. But, it is worth highlighting here that, in contrast to the "round brackets", the "square ones" are used in Ф+∃ not as the pure technical (auxiliary) symbols, but as *ontologically meaningful* signs. Such very odd (unusual, unhabitual) utilizing the "square brackets" is a psychological surprise (difficulty), because, in relation to natural language, round brackets and square ones seem identical and very often are used (in natural language) as synonyms. But, in contrast to natural language, in the artificial object-language of Ф+∃, the two kinds of brackets have *substantially different* meanings (play *significantly different* roles): exploitation of round brackets is purely technical (auxiliary) one, on the contrary, square-bracketing have an *ontological* meaning. The *ontological* meaning of the nonstandard usage of square-brackets is precisely defined below in that part of the given article which part deals with *semantics* of object-language of Ф+∃. However, even at the level of pure syntaxis of the object-language of Ф+∃, square brackets *play an important role* in giving exact definition of/for the notion "formula of Ф+∃". (Such exact definition is to be provided below in just this very section of the paper.) Moreover, the nonstandard usage of square-brackets *plays a significant role* also in the exact formulations of some axiom-schemes of Ф+∃" (which formulations are to be provided below also in just this very section of the paper).

10. A very strange complex sign "=+=" (artificially composed of the habitual signs) belongs to the alphabet of object-language of Ф+∃. The sign "=+=" is called (a symbol of) "*formal-axiological equivalence*". The very odd compound symbol "=+=" *plays a very important role* in the exact definition of "formula of Ф+∃" and also in the exact formulations of some axiom-schemes of Ф+∃.

11. A sign belongs to the alphabet of object-language of Ф+∃, then and only then, when the sign is an element of this alphabet owing to the above-located points 1) – 10) of the given definition.

Any finite sequence (train) of symbols is called "an *expression* of the object-language of Ф+∃", if and only if that sequence contains such and only such symbols which belong to the alphabet of object-language of Ф+∃.

An exact definition of the notion "*term* of Ф+∃" is the following.

1. The above-listed *axiological variables* (see the above-located definition of alphabet of Ф+∃) are terms of Ф+∃.

2. The above-indicated *axiological constants* (see the above-provided definition of alphabet of Ф+∃) are terms of Ф+∃.

3. If $Ф_k^n$ is some *n-placed value-functional symbol* (mentioned in the above-located definition of alphabet of Ф+∃), and $t_i, \ldots t_n$ are *terms* of Ф+∃, then any expression possessing the form $Ф_k^n t_i, \ldots t_n$ is a term of Ф+∃. (Here, it is worth taking into an account that symbols $t_i, \ldots t_n$ belong to the meta-language because they denote *any* terms of Ф+∃; the similar remark is worth making in connection with the symbol $Ф_k^n$, which also belongs to the meta-language of Ф+∃.)

4. Any expression, belonging to the object-language of Ф+∃, is a term of Ф+∃, if and only if this is so owing to the above-formulated points 1) – 3) of the present definition.

Thus, the purely *syntactic* aspect of the abstract concept "*term* of Ф+∃" is perfectly fixed. Therefore, now it is quite opportune to go to precise defining the purely *syntactic* aspect of the abstract concept "*formula* of Ф+∃". To perform this correctly, let us accept the agreement that in the present paper, lowercase Greek letters α, β, and ω (which belong to meta-language of Φ+∃) denote *any* formulae of Φ+∃. Keeping this agreement in mind, it is possible to construct the below-located exact definition of the concept "formula of Φ+∃".

1. All such lowercase Latin letters which are named "*dictum variables*", and also all such lowercase Latin letters which are named "*dictum constants*", are elements of the set of formulae of Φ+∃.
2. If α and β are formulae of Φ+∃, then all such expressions of the object-language of Φ+∃, which (expressions) possess the logic-forms (α & β), (α ∨ β), (α ⊃ β), (α ↔ β), ¬α, are elements of the set of formulae of Φ+∃ as well.
3. If t_i and t_k are terms of Φ+∃, then (t_i =+= t_k) is a formula of Φ+∃.
4. If t_i is a term of Φ+∃, then [t_i] is a formula of Φ+∃.
5. If α is a formula of Φ+∃, and the sign Ψ (which belongs to the meta-language of Φ+∃) denotes any modality-symbol belonging to the set {□, Ж, K, A, E, S, T, F, P, D, C, Y, G, W, O, B, U, J}, then all such expressions of object-language of Φ+∃, which possess the form Ψα, are formulae of Φ+∃. Here, it is worth keeping in mind, that, strictly speaking, the expression having form Ψα is not a formula of Φ+∃, but a scheme of formulae of Φ+∃.
6. Finite successions (trains, tails) of signs belonging to the alphabet of object-language of Φ+∃ are formulae of Φ+∃, when and only when this is so owing to the points 1) – 5) of the present definition.

This special part of the paper is deliberately reduced exclusively to *syntactic* meanings of expressions of object-language of the formal axiomatic philosophy theory Φ+∃. Therefore, the set of modal symbols {□, Ж, C, Y, G, W, K, E, A, S, O, B, U, J, T, F, P, D} is considered here as nothing but a set of extremely short *names*. The sign □ is a name for the well-known modality "it is *necessary* that…" The sign (Cyrillic letter) Ж is a name of/for the *newly* introduced philosophical-*ontology* modality "what is indicated-and-described by the *dictum* (affirmation) …, *exists*". Thus, the *ontological* modality Ж is a *de dicto* modality. According to the tradition, modalities *de-dicto* are glued to a *dictum*. It is a statistical norm that, from the Latin language, the word "*dictum*" is translated as an "assertion, or affirmation (or statement, or sentence, or judgement)". However, in principle, there is such a heuristically important possibility of substantial *generalization* of the traditional meaning of the word "*dictum*", according to which (generalization) any theoretical (deductive) system may be also considered as a *dictum*. Thus, by the substantially generalized definition accepted and used systematically in this article, (in general) dictum is either a proposition or a theory. According to its traditional meaning, dictum is "what is affirmed (asserted, stated)", but a proper theory also can be considered as "what is affirmed (asserted, stated)". Here it is worth emphasizing that the sign Ж (standing for the ontological modality *de dicto*) does not belong to the alphabets of artificial languages of Σ, Σ+C, Σ-2C, and Φ. The theory Φ+∃ is a result of adding the sign Ж to the set of modality symbols belonging to the alphabet of object-language of Φ.

The signs K, E, A, S, T, F, P, D, Y, C, respectively, stand for the modal expressions "person *Knows* that…", "person *Empirically* (*a-posteriori*) *knows* that…", "person *A-priori knows* that…", "in some fixed time-and-space, under some special conditions, a person has a *Sensation* (either by means of some instruments, or immediately), that…", "it is *True* that…", "person has *Faith* that… (or person believes that…)", "in a consistent theory, it is *Provable* that…", "*an algorithm* exists *for Deciding* that… (consequently, there is a possibility of/for constructing a machine for such Deciding)", "it is *Complete* that …", "it is *Consistent* that …"

The signs G, W, O, B, U, J, respectively, stand for the modal expressions "it is *Good* (perfect in moral sense) that…", "it is *Wicked* (bad in moral sense) that…", "it is *Obligatory* (or it is a duty) that …", "it is *Beautiful* (perfect in aesthetic sense) that …", "it is *Useful* (beneficial) that …", "it is a *Joy* (mirth, pleasure, gladness, happiness, sunshine) that …". In this special part of the paper, purely syntactic meanings of the modal symbols are introduced and defined precisely by the below-provided schemes of own (proper) formal-philosophy axioms of the multimodal epistemology-and-axiology-and-ontology system Ф+∃. Certainly, the mentioned axiomatic definition is indirect (not manifest) one, but, nonetheless, it is quite precise and sufficient for rational philosophizing.

In this formal philosophy system, the proper (own) axioms of philosophical *ontology*, universal epistemology, and formal axiology are added to pure formal-logic axioms which (logic-axioms) are evidently *similar* (analogous) to the ones of classical formal logic of propositions. Thus, pure formal-logic axioms and formal-logic-inference rules of Σ, Σ+C, Σ-2C, and Ф+∃ are significantly *analogous* to the ones of classical propositional logic calculus. Certainly, the reference to the essential *similarity* is not a perfect definition, therefore, below I define the set of pure formal logic axioms of Ф+∃ rigorously by the following schemes of formulae SLA-1, SLA-2, SLA-3. If α, β, and ω are formulae of Ф+∃, then the below-placed schemes of formulae of Ф+∃ are schemes of pure logic axioms of Ф+∃. (Here "SLA" is an abbreviation of "sentential logic axiom").

SLA-1: $\alpha \supset (\beta \supset \alpha)$.

SLA-2: $(\alpha \supset (\beta \supset \omega)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \omega))$.

SLA-3: $(\neg\alpha \supset \beta) \supset ((\neg\alpha \supset \neg\beta) \supset \alpha)$.

By definition, Ф+∃ contains only one formal-logic derivation-rule, namely, "MP (*modus ponens*)". It is formulated as usually: if α and β are formulae of Ф+∃, then $\alpha, (\alpha \supset \beta) \vdash \beta$. (As usually, the symbol " $\ldots \vdash \ldots$" stands for "in Ф+∃, from … it is formally-logically derivable that…".) In spite of the fact that, in the definition of Ф+∃, only one formal-logic inference-rule is mentioned, there is possibility deductively to derive and systematically to exploit also *additional* logic-inference-rules formally derivable in the classical sentential logic.

It is worthy of being highlighted here that, strictly speaking, SLA-1, SLA-2, SLA-3, are outcomes of a significant *innovation* (substantially helpful one) in the classical sentential-logic axioms; the above-introduced *dictum variables* have replaced the corresponding sentential ones. As to the habitual definitions of the well-known additional logic-connectives, and the habitual logic-inference-rules (*modus ponens* and all handy derivative ones), the mentioned innovation (generalization) is to be kept in mind as well.

When I talk of the additional logic-connectives, I mean the logic-connectives represented by symbols &, ∨, and ↔, which are used systematically in the below-submitted axiom-schemes of Ф+∃ and in the formal inferences (in Ф+∃) under construction and discussion in this article. Above, only ¬ and ⊃ are defined by SLA-1, SLA-2, SLA-3, although ∨, &, and ↔ are used in this paper as well. Therefore, strictly speaking, here-now, it is indispensable accurately to determine meanings of $\vee$, &, and $\leftrightarrow$, by the following definitions Def(a), Def(b), Def(c), respectively.

Def(a): $(\alpha \vee \beta)$ stands for $((\neg\alpha) \supset \beta)$.

Def(b): $(\alpha \,\&\, \beta)$ stands for $(\neg((\neg\alpha) \vee \neg\beta))$.

Def(c): $(\alpha \leftrightarrow \beta)$ stands for $((\alpha \supset \beta) \,\&\, (\beta \supset \alpha))$.

The purely-logical axiom-schemes SLA-1, SLA-2, SLA-3, the relevant definitions of supplementary logic connectives, and the logic-derivation-rules (the famous "*modus ponens*" and all possible *derivative* logic-inference-rules) are applicable to all formulae of formal systems Σ, Σ+C, Σ-2C, Φ, and Φ+∃. Hence, the formal-logical foundations of Σ, Σ+C, Σ-2C, Φ, and Φ+∃ are absolutely identical but the mentioned formal axiomatic systems grounded upon these utterly identical pure-logic foundations are qualitatively different. It actually *seems* that, corresponding definitions of formal theories Σ, Σ+C, Σ-2C, Φ, and Φ+∃ are absolutely identical, but it *only seems* so, as the absolute identity is a truthlike illusion. The impression of absolute identity of the formal theories is a gravely misleading mistake because, strictly speaking, Σ, Σ+C, Σ-2C, Φ, and Φ+∃ have different object-language alphabets and, consequently, different sets of expressions (of the artificial languages), different sets of terms (and, consequently, different sets of formulae), different sets of proper axioms, and, consequently, different sets of theorems. Thus, even at the level of syntax, Σ, Σ+C, Σ-2C, Φ, and Φ+∃ are qualitatively different formal theories.

In the present section of the paper, the *syntactic* meanings of the modality signs and of the other symbols belonging to the alphabet of object-language of Φ+∃ are defined precisely by the below-located list (AX-1 – AX-11) of schemes of proper-philosophy (ontology, epistemology, axiology) axioms of Φ+∃. (Obviously, the *axiomatic* definition of proper-philosophy notions, namely, universal epistemological, ontological and axiological concepts is *not a manifest* definition. Notwithstanding, it is *perfectly exact* one.) When α, β, ω are formulae of Φ+∃, then all such (and only such) expressions of the object-language of Φ+∃, which have the following logic forms, are *proper-axioms* of Φ+∃.

Axiom scheme AX-1: $A\alpha \supset (\Omega\beta \supset \beta)$. It is worth emphasizing here that, in Φ+∃, AX-1 is significantly more universal one than in Σ, Σ+C, and Σ+2C.

Axiom scheme AX-2: $A\alpha \supset (\Omega(\omega \supset \beta) \supset (\Omega\omega \supset \Omega\beta))$. Also, it is worth emphasizing here that, in Φ+∃, AX-2 is significantly more universal one than in Σ, Σ+C, and Σ+2C.

Axiom scheme AX-3: $A\alpha \leftrightarrow (K\alpha \,\&\, (\neg\Diamond\neg\alpha \,\&\, \neg\Diamond S\alpha \,\&\, \Box(\beta \leftrightarrow \Omega\beta)))$.

Axiom scheme AX-4: $E\alpha \leftrightarrow (K\alpha \,\&\, (\Diamond\neg\alpha \vee \Diamond S\alpha \vee \neg\Box(\beta \leftrightarrow \Omega\beta)))$.

Axiom scheme AX-5: $\Omega\alpha \supset \Diamond\alpha$. (This is a significant *generalization* of the so-called "Kant principle" linking the deontic modality with the alethic one: $O\alpha \supset \Diamond\alpha$).

Axiom scheme AX-6: $(\Box\beta \,\&\, \Box\Omega\beta) \supset \beta$. (This is a significant *generalization* of the very important formula $(\Box\beta \supset \beta)$, which is logically underivable in Φ+∃.

Axiom scheme AX-7: $(t_i =+= t_k) \leftrightarrow (G[t_i] \leftrightarrow G[t_k])$.

Axiom scheme AX-8: $(t_i =+= g) \supset \Box G[t_i]$.

Axiom scheme AX-9: $(t_i =+= b) \supset \Box W[t_i]$.

Axiom scheme AX-10: $(G\alpha \supset \neg W\alpha)$.

Axiom scheme AX-11: (Wα ⊃ ¬Gα).

Definition scheme DF-1: when ω is a formula of Φ+∃, then ◊ω is a *name* of/for ¬□¬ω.

In AX-1, AX-2, AX-3, AX-4, AX-5, and AX-6, the sign Ω (belonging to the meta-language of Φ+∃) denotes a (any) "*perfection* modality" exclusively. Not all the above-listed modalities are called "*perfection* ones". (One may call them just "perfections".) The set Δ of signs standing for *perfection*-modalities is the following {Ж, K, D, F, C, Y, P, J, T, B, G, U, O, □}. Evidently, Δ is only a subset of the above-listed set of all symbols denoting modalities under discussion in this paper. For example, W and ◊ are signs of such modalities, which are not perfections.

Including Ж into the set Δ of *perfection*-modalities is quite natural as, certainly, "being (existence)" is an important *perfection*. For instance, "being (existence)" is an essential *perfection* (and one of the *names*) of God [Corrigan, Harrington, 2023; Ершов, Самохвалов, 2007, pp. 118-131; Descartes, 1970; Hartshorne, 1962, 1965; Janowitz, 1991; Logan, 2009; Nolan, 2021; Oppenheimer, Zalta, 1991; Oppy, 2023; Parsons, 1980; Pseudo-Dionysius Areopagite, 1980; Reicher, 2022; Williams, 2023]. That is why some well-known *ontological* proofs of God's *existence* have been based upon the general presumption that what does not exist, is not perfect. Including modal symbols C and Y into the set Δ (composed of *perfection*-modalities exclusively) is also quite reasonable as "consistency" and "completeness" are very important *perfections* of any proper theoretical system [Тарский, 1948, с. 185-186]. The inclusion of "consistency" and "completeness" into the set of *perfections*, is in accordance (harmony) with the quite adequate and well-demonstrated general idea of *normativeness* of deductive discourse and, in particular, of treating consistency and completeness as the *standards (norms)* of deductive thinking [Целищев, 2004а, 2004б, 2005].

From the viewpoint of *formal modal logic* of values, preferences, and assessments, the axiom schemes AX-10 and AX-11 are quite clear and obvious. In contrast to them, the almost unknown (extraordinary, aunhabitual) nontrivial axiom-schemes AX-7, AX-8, AX-9 represent not the symbolic *formal logic* of evaluative modalities but a symbolic *formal axiology* – general theory of abstract-value-forms of any (either existing or not-existing) things. (This is an option of systematical rationalizing Meinongianism, or a special kind of its being quite consistent.) The concept "symbolic *formal-logic*" is not identical (logically) to the concept "symbolic *formal-axiology*", hence, "formal-logic inconsistency" and "formal-axiological inconsistency" are not synonyms.

Obviously, in any concrete relation to that world, which is external to Φ+∃, the above-submitted precise *syntactic* definitions make no sense; they are *semantically meaningless*. However, this is not a defolt (delinquency) committed by negligence; this is a quite consciously accepted scientific abstraction. The deliberately established (allowed) scientific abstraction is quite resonable, if and only if, the realm of its adequateness is well-defined. Hence, for making this paper quite meaningful one, I am to move now from the above-defined *syntax* to a hitherto not defined *semantics* of the artificial language of Φ+∃.

2.2. Defining semantics of the formal axiomatic philosophy system Ф+∃ synthesizing ontology with epistemology and axiology

The above-placed section 2.1 of this paper, presents the *purely syntactic* definition of Ф+∃ which has been intentionally deprived of its relevant philosophical contents (due to the accepted scientific abstraction). The formal-philosophy axiomatic system Ф+∃ is a *multimodal* one, but hitherto concrete contents of the modalities under consideration have been revealed not sufficiently; the theory Ф+∃ has been considered as an exactly *formal* theory. Now, in the given part of the paper, namely, in the section 2.2, I am to relax the formality of Ф+∃ by shifting immediately to *concrete philosophical contents* of the above-mentioned modalities studied in Ф+∃. In the present paper it is implied that semantic meanings of the habitual artificial-language signs of classical symbolic logic are already introduced and well-defined owing to relevant handbooks. As the quite clear semantic meanings of the relevant proper-logic symbols are well-known, it is redundant to define them here. But, such unusual (inhabitual, perhaps, very odd) signs of the artificial language of Ф+∃, which are exploited systematically in the proper-philosophy-axioms (ones of epistemology, ontology, etc.), require special introduction and precise definition of their semantic meanings.

Meanings of the lowercase Latin letters q, p, d (and of the same letters possessing lower number indexes) named "*dictum-variables*" are analogous to the meanings of the habitual "*propositional variables*". But there is a substantial difference: in Ф+∃, values of "*dictum variables*" belong to the set of *dictums*, to which (set) not only all true or false *sentences* (statements) but also all true or false theories (logically organized systems of propositions) belong. Thus, generally speaking, the dictum-variables range over the set of either true or false dictums. If an interpretation of Ф+∃ is provided (well-defined), then a *dictum-constant* means (in the given interpretation) quite a definite (perfectly fixed) element from the set of dictums, namely, either a concrete true or false sentence (statement) or a concrete true or false theory.

According to the habital (statistical) linguistic norm (custom rule), from the Latin language, "*dictum*" is to be translated as "an expression of (a thought …) in words", for example, as "a proposition (sentence) *q*", or "an affirmation of (…)". But, there is a heuristically important possibility deliberately to shift from "affirmation of (a proposition …)" to a *significantly more general* "affirmation of (a proposition …, or a theory …)" as along with uttering separate statements, one can affirm also a theory (logically organized system of statements). As the indicated innovative generalization is accepted, in this paper, it is presumed (as a hypothesis wothy of investigation) that a theory is also a dictum. Hence, attaching *de-dicto*-modalities Ж (existence), C (consistency), T (truth), Y (completeness), and D (decidability) to theories is vindicated in Ф+∃. Relevant information of modalities *de-dicto* and *de-re,* along with interesting philosophical discussing their interconnections, can be found, for example, in [Kneale, 1962; Целищев, 1978].

Defining semantic meanings of formal-language-expressions is defining an *interpretation-function*. For defining the interpretation-function, it is necessary to define precisely: (1) such a set which is called "realm (or domain) of interpretation" (hereafter the letter M denotes the set which is the domain of interpretation); (2) an "assessor (valuator)" *V*. If a standard interpretation of Ф+∃

is fixed, then, by definition, M is such a set, each element of which possesses: (1) one and only one proper *axiological value* belonging to the set {good, bad}, and (2) one and only one proper *ontological value* belonging to the set {exists, not-exists}.

The *axiological variables* (x, y, z, x_k, y_m, z_i, …) take their values from the domain of interpretation (M).

The *axiological constants* "b" and "g" denote abstract *axiological values* "bad" and "good", respectively.

Valuating an element belonging to M by a definite (fixed) assessor V is nothing but ascribing an *axiological value* to the element. The assessor V may be either an individual or a collective (it does not matter). Certainly, any change of V can result in a change of some (relative) evaluations, nevertheless, such mutations cannot change the set of absolutely immutable formal-axiological laws of two-valued algebra of metaphysics (as formal axiology), which absolutely universal laws are not relative but absolute moral evaluations. The laws in question are *constant valuation-functions* possessing the axiological value "good" under any combination of the values of their arguments. Although V is such a *variable*, values of which belong to the set of all possible assessors (interpreters), any well-defined interpretation of Φ+∃ necessarily implies that the value of assessor-*variable* V is fixed. Any change of value of V means a change of interpretation.

In the given paper, "e" and "n" denote "… exists" and "… not-exists", respectively. The lowercase Latin letters "e" and "n" are called "*ontological constants*". In any standard interpretation of Φ+∃, by definition, one and only one element of the four-element-ed set {{g, n}, {g, e}, {b, n}, {b, e}} corresponds to every element of the domain of interpretation (the above-introduced set M). That is why Φ+∃ may be considered as a formal semantic representation (discrete mathematical model) of an important truth existing in "Meinong's jungles" [Meinong, 1960; Russell, 1905, 1941, 1992; Jacquette, 1996, 1997, 2015; Marek, 2022; Parsons, 1980, 1982; Perszyk, 1993; Berto, 2012; Berto and Priest, 2014; Routley, 1980; Zalta, 1983]. The lowercase Latin letters "e" and "n" belong to the alphabet of meta-language. They do not belong to the alphabet of object-language of Φ+∃, according to the above-given definition (of the alphabet). Notwithstanding, "e" and "n" are represented in the object-language of Φ+∃ *indirectly* by means of *square-bracketing*: the *ontological* statement-form "t_i exists" is represented by formula [t_i]; the *ontological* statement-form "t_i does not exist" is represented by formula ¬[t_i]. This implies that square-bracketing is a very important aspect of precise defining the proper philosophical *semantics* of Φ+∃.

In my opinion, the above-said (of "t_i exists" and "t_i does not exist") is an adequate formal semantic representation of the extraordinary doctrine uniting existent and nonexistent objects in one system of philosophical ontology [Fine, 1984, 1985; Hintikka, 1984; Jacquette, 1996, 1997, 2015; Marek, 2022; Parsons, 1980, 1982; Perszyk, 1993; Priest, 2005; Reicher, 2022; Smith, 1985; Zalta, 1983].

N-placed terms of Φ+∃ are interpreted as *n*-placed evaluation-functions defined on the set M. The notion "one-placed evaluation-function" is exemplified below by the Table 1. (It is relevant to recall here that the upper index 1 standing immediately after a capital letter means that this letter stands for a one-placed evaluation-function).

Table 1

Definition of the evaluation-functions determined by one evaluation-argument

x	B_1^1x	N_1^1x	U_1^1x	A_1^1x	C_1^1x	I_1^1x	Z_1^1x	S_1^1x	G_1^1x	P_1^1x	H_1^1x	C_2^1x	I_2^1x
g	g	b	b	g	g	b	b	b	g	g	b	g	b
b	b	g	b	g	b	g	b	b	g	b	g	b	g

In the Table 1, the one-placed term B_1^1x is interpreted as *one-placed evaluation-function* "*being (existence) of* (what, whom) *x*"; the term N_1^1x is interpreted as *evaluation-function* "*non-being (nonexistence) of* (what, whom) *x*". U_1^1x – "*absolute non-being of* (what, whom) *x*". A_1^1x – "*absolute being of* (what, whom) *x*". C_1^1x – "*consistency of* (what, whom) *x*", or "*x's being consistent*". I_1^1x - "*inconsistency of* (what, whom) *x*", or "*x's being inconsistent*". Z_1^1x – "*a formal-axiological contradiction* (what, who) *x*", or "*x's being absolutely inconsistent*". S_1^1x – "*x*'s *formal-axiological self-contradiction*". G_1^1x – "*absolute goodness of* (what, whom) *x*", or "*absolutely good* (what, who) *x*". P_1^1x – "*positive evaluation of* (what, whom) *x*". H_1^1x – "*negative evaluation of* (what, whom) *x*". C_2^1x - "*completeness of* (what, whom) *x*". I_2^1x – "*incompleteness of* (what, whom) *x*".

The notion "*two-placed evaluation-function*" is instantiated by the below Table 2. (In this paper, the upper index 2 standing immediately after a capital letter means that this letter stands for *a two-placed function*; difference of lower number-indexes means difference of the relevant symbols, for example, in the above-presented Table 1, C_1^1x and C_2^1x are different symbols.)

Table 2

Definition of the evaluation-functions determined by two arguments

x	y	K^2xy	S^2xy	X^2xy	T^2xy	Z^2xy	P^2xy	C^2xy	E^2xy	V^2xy	N^2xy	Y^2xy
g	g	g	b	b	b	b	g	g	g	b	b	g
g	b	b	g	b	b	b	g	b	b	g	b	g
b	g	b	g	g	g	g	b	g	b	g	b	g
b	b	b	g	b	b	b	g	g	g	b	g	b

In the Table 2, the two-placed term K^2xy is interpreted as evaluation-function "*being of both x* and *y* together", or "*joint being* of *x* with *y*". S^2xy is interpreted as "*separation, divorcement* between *x* and *y*. The term X^2xy – evaluation-function "*y's being without x*", or "joint being of *y* with nonbeing of *x*". T^2xy – "*termination* of *x* by *y*". Z^2xy – "*y*'s *contradiction* to (with) *x*". P^2xy - "*preservation, conservation, protection* of *x* by *y*". C^2xy is interpreted as evaluation-function "*y's existence, presence in x*". E^2xy – "*equivalence, identity (of values)* of *x* and *y*". V^2xy – "*choosing and realizing* such and *only such* an element of the set {*x, y*}, which is: 1) *the best* one, if both *x* and *y* are good; 2) the least bad one, if both *x* and *y* are bad; 3) the good one, if *x* and y have opposite values. (Thus, V^2xy means an *excluding choice and realization of only the optimal* between *x* and *y*.) The term N^2xy is interpreted as evaluation-function "*realizing neither x nor y*". Y^2xy is interpreted as evaluation-function "realizing a *not-excluding-choice* result, i.e. 1) *realizing* K^2xy *if both x and y are good*, and 2) *realizing* V^2xy *otherwise*". Additional exemplifications of "*two-placed evaluation-function*" may be found in [Lobovikov, 2020, 2021, 2022, 2023].

To avoid possibile misunderstanding this article, now it's timely to emphasize that in a standard interpretation of Ф+∃, the symbols V^2xy, C^2xy, K^2xy, E^2xy, $C_1{}^1x$, $B_1{}^1x$, $N_1{}^1x$ stand not for some predicates but for some *n*-placed *evaluation-functions*. If an interpretation of Ф+∃ is given, then the expressions of object-language of Ф+∃, which (expressions) possess the forms (t_i=+=b), (t_i=+=g), (t_i=+=t_k), are representations of some *predicates* in Ф+∃.

According to the definition of semantics of Ф+∃, if t_i is a term of Ф+∃, then, if a formula of Ф+∃, possessing the form [t_i], is interpreted, then the mentioned formula represents (in the given interpretation) an *either false or true proposition* having the form "t_i exists". Thus, according to the definition, in any standard interpretation, any formula [t_i] is true then and only then, when t_i possesses the *ontological value* "e (exists)" in the given interpretation. Also, in any standard interpretation of Ф+∃, any formula [t_i] is false, then and only then, when t_i possesses the ontological value "n (not-exists)" in the given interpretation.

By definition of semantics of Ф+∃, in a standard interpretation of Ф+∃, the formula scheme (t_i =+= t_k) is a proposition possessing the form "t_i is *formally-axiologically equivalent* to t_k"; this proposition is true if and only if (in that interpretation) the terms t_i and t_k obtain identical *axiological values* (from the set {good, bad}) under any possible combination of *axiological values* of their *axiological variables*.

By definition of semantics of Ф+∃, in a standard interpretation of Ф+∃, the formula scheme (t_i =+= b) is a proposition having the form "t_i is a *formal-axiological contradiction*" (or "t_i is *formally-axiologically, or invariantly, or absolutely bad*"); this proposition is true if and only if (in that interpretation) the term t_i acquires axiological value "bad" under any possible combination of axiological values of the axiological variables.

By definition of semantics of Ф+∃, in a standard interpretation of Ф+∃, the formula scheme (t_i =+= g) is a proposition having the form "t_i is a *formal-axiological law*" (or "t_i is *formally-axiologically, or invariantly, or absolutely good*"); this proposition is true if and only if (in the interpretation) the term t_i acquires *axiological value* "good" under any possible combination of axiological values of the axiological variables.

In respect to the above-given definition of sematic meaning of (t_i =+= t_k) in Ф+∃, it is indispensable to highlight the important linguistic fact of homonymy of the words "is", "means", "implies", "entails", "equivalence" in natural language. On the one hand, in natural language, these words may have the well-known formal logic meanings. On the other hand, in natural language, the same words may stand for the above-defined *formal-axiological-equivalence* relation "=+=". This ambiguity of natural language is to be taken into an account; the different meanings of the homonyms are to be separated systematically; otherwise the homonymy can head to logic-linguistic illusions of paradoxes.

Due to the above-given definition of proper philosophical semantics (formal-axiological-and-ontological one) of/for the formal theory Ф+∃, readers can easyly recognize that the above-formulated two-valued algebraic system of formal axiology plays the role of such abstract *theory-*

of-relativity of evaluations, in which (relativity theory), the *laws* (*formal-axiological* ones) of that algebraic system are nothing but constantly-good evaluation-functions. In other words, the absolutely universal and immutable evaluation-relativity laws are *invariants* with respect to all possible transformations of assessor (interpreter) V.

Thus, in spite of the obvious fact that relativity and impermanence (volatility) of proper-empirical valuations does exist in the sensory material world, the valuation-invariants (which are absolutely immutable universal laws of valuation-relativity) also do exist [Lobovikov, 2020].

2.3. Conditions of Truth of Formulae of the Theory Ф+∃ in the Standard Model of It

In relation to the nontrivial problem (sharply formulated in the radical logic empiricism, for instance, by A. Ayer) of proper logic (truth-related) status of moral *evaluations* – judgements of moral *value* (which are *either good or bad moral acts* in accordance with the two-valued *algebra of formal ethics*), it is worth attracting special attention to the formal-*axiological equivalences* (B_1^1x =+= x) and (P_1^1x =+= x), where meanings of *terms* x, B_1^1x, and P_1^1x belong to the above-defined set M (called "domain of interpretation"), every element of which acquires one and only one of the four values which are elements of the four-element-ed set of two-element-ed sets {{g, e}, {g, n}, {b, n}, {b, e}}. (Perhaps, here it is worth reminding that: "g" means "good"; "e" means "exists"; "n" means "not exists"; "b" means "bad".)

From the viewpoint of the above-said, it is easy to see that the formal-*axiological equivalences* (P_1^1x =+= x) and (B_1^1x =+= x) are *similar* to (or are *analogues* of/for) the famous formal-*logical equivalence* ($Tp \equiv p$), systematically discussed in A. Tarsky logic semantics. However, in this connection, it is important to keep in mind that, generally speaking, *similarity* (*analogy*) *is not an identity*. Generally speaking, analogousness (similarity) relation is not transitive, consequently, it is not an equivalence relation. According to A. Ayer, M. Schlick, and many other representatives of logic positivism, in *logic as truth theory*, strictly speaking, *judgements of values* (moral, aesthetic, religious, etc.) *are neither true nor false* [Айер, 2010, с. 35-38, 147-172], consequently, from the viewpoint of *proper logic* semantics, they are *meaningless* [Там же, с. 35-38, 45-63, 102-124, 147-172].

However, the logically formalized theory Ф+∃ is not a *pure logic* system, i.e. *only* logic and nothing more than logic. Ф+∃ is an outcome of *application* of logic to what is not logic but has *own proper* axioms (epistemological, axiological, et al).

Moreover, the theory Ф+∃ is *multimodal* one (truth and falsity are placed in it among many other qualitatively different species of modalities, and are considered along with the modalities of *values* (for instance, ethic *goodness*, *wickedness*, and other ones).

To exclude possible confusions and misunderstandings of the following, it is worth highlighting here that "t" without indexes stands in this article for "true", while "t" with a lower literal index stands for a *term*.

In semantics of the language of Ф+∃, truth of moral *assessment* (i.e. judgement of moral *value*) $G[t_i]$ is defined precisely by the *formal-logical tantamount-ness*: (($G[t_i]$ acquires *logic* value "t") ≡ (term t_i acquires *axiological* value "g")). Hereafter, the symbol "≡" stands for the well-known semantic relation of *formal-logical equivalence* (coincidence of logic values). The above-provided *formal-logical tantamount-ness* (playing the role of fundamental *definition* accepted in this article) explicates (precisely determines) semantic meaning of the modal expression *G*[t_i] of the artificial language of Ф+∃.

Taking into account all the above-said, conditions of truth of formulae in standard model of formal theory Ф+∃ are defined in this article as follows:

I. ((formula [t_i] has *logic* value "t") ≡ (term t_i has *ontological* value "e")).

II. ((formula *G*[t_i] has *logic* value "t") ≡ (term t_i has *axiological* value "g")).

III. ((formula *W*[t_i] has *logic* value "t") ≡ ⌉(term t_i has *axiological* value "g")).

IV. For any formula α, formula $K\alpha$ has *logic* value "t", if and only if, either it is true that $E\alpha$, or it is true that $A\alpha$. Here, the connective "either …, or …" denotes the strict (exclusive) disjunction. Hence, there is a possibility to utilize the below-provided hexagon containing the square of opposition for graphic modelling (visualizing) the system of semantic (truth-related) logical interconnections among the *epistemic* modalities *K*, *A*, *E*, while defining conditions of truth of formulae of the theory Ф+∃ in a standard model of it.

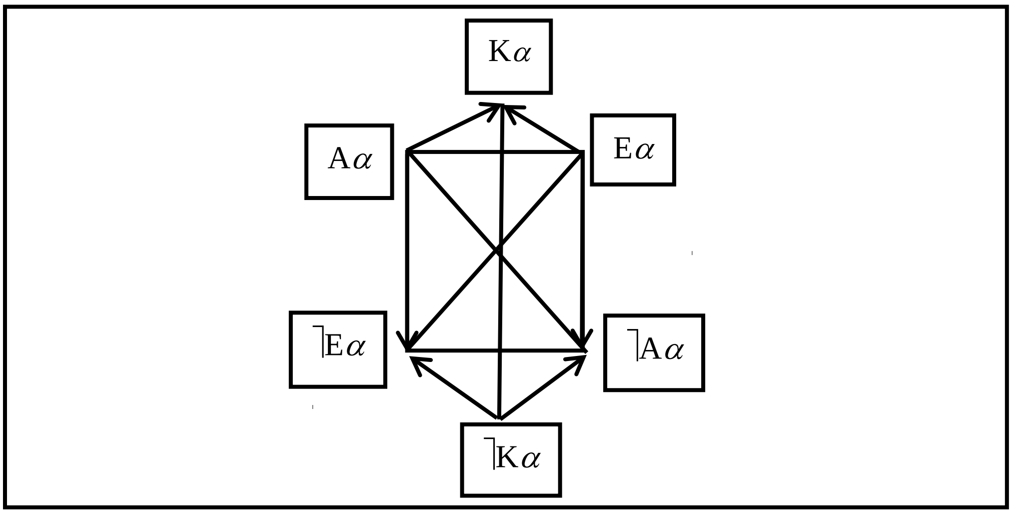

Fig. 1. The hexagon and logical square of opposition of epistemic modalities

In perfect accordance with the well-known traditional formal logic, in Fig. 1, the upper horizontal line models graphically the *contrariety* relation; the bottom horizontal line models graphically the *sub-contrariety* relation; the lines crossing the square represent graphically the *contradictoriness* relations. The *logic consequence* relations are visually modeled by arrows. In relation to the epistemic modalities, all the formal-logic semantic (truth-related) rules of the traditional formal-logic-square are valid. The pedagogically and heuristically fruitful idea of graphic modelling proper logic aspect of abstract concept systems by means of the logic square and hexagon has been known and used systematically since ancient times; nowadays, plenty of *qualitatively new* interpretations of this old idea are invented and exploited. Its above-provided *epistemic modality interpretation*, namely, Fig. 1 has been suggested originally in [Лобовиков, 2014, с. 68; Лобовиков, 2016, с. 50].

According to the present article, in a standard model of formal theory Ф+∃, conditions of truth of/for such formulae, which contain the expression $E\alpha$, are defined as follows:

V. For any formula α, if it is true that $E\alpha$, then it is true that $K\alpha$.

VI. For any formula α, if it is true that $\Diamond S\alpha$, then it is true that $E\alpha$. This is a representation (in semantics of Ф+∃) of the requirement of positivist methodology of science, according to which (requirement), proper *empirical* knowledge must be sensually *verifiable at least in principle* [Айер, 2010, с. 13-29, 48-52]. It is easy to notice that the given representation (modelling) of verification-ism significantly differs from the original (here I imply the significant difference between *necessary* and *sufficient* conditions of/for *empirical-ness* of knowledge).

VII. For any formula α, if it is true that $\Diamond\rceil\alpha$, then it is true that $E\alpha$. This is a representation (in semantics of Ф+∃) of the principle of methodology of science (condition of/for scientific-ness), according to which (principle), proper scientific (*empirical*) knowledge must be *falsifiable at least in principle* [Айер, 2010, с. 52]. It is easy to see that the given *representation (modelling)* of falsification-ism significantly differs from the original (I mean the essential difference between *necessary* and *sufficient* conditions of/for *empirical-ness* of knowledge, as it has been already mentioned in the previous item VI). K. Popper has paid very special attention to fundamental falsifiability of scientific knowledge [Поппер, 1983, с. 105-123]. In this connection, O. Neurath [Neurath, 1982, pp. 121-131], A. Айер [Айер, 2010, с. 52-53], and some other celebrated specialists in philosophy of science have criticized Popper for an overly obsessive (pseudo-rationalistic) desire to oust and replace the criterion of fundamental sensual verifiability with the criterion of fundamental falsifiability, in methodology of science. Neurath has rightly pointed out that, being *sufficient* conditions of/for scientific-ness (empirical-ness), neither verifiability, nor falsifiability, nor their non-excluding disjunction are *necessary* conditions of/for proper scientific (empirical) knowledge; there are also some other not-well-recognized nontrivial criterions of scientific-ness (empirical-ness) of knowledge. Which criterions are implicitly meant by Neurath? Unfortunately, he has given no quite definite answer to this naturally arising question; even no guess in this connection has been formulated. Nonetheless, in the present article, the unclear and implicit Neurath's abstract intuition is clarified and represented (modelled) manifestly by the following concrete statement VIII, which is also a *sufficient* (but not necessary) condition of/for empirical-ness of knowledge.

VIII. For any formulae α and ω, if it is possible that logic values of expressions (formula $\Omega\omega$ has logic value "t") and (formula ω has logic value "t") are different, then it is true that $E\alpha$. Perhaps, it is worth reminding here that symbol Ω denotes a (any) element of the set of perfection modalities (this set is defined above). The *necessary and sufficient* condition of *empirical-ness* of knowledge (in standard model of formal theory Ф+∃) is the following condition IX.

IX. For any formulae α and ω, if and only if, in a standard model of the formal theory Ф+∃, it is true that $E\alpha$, then, in the standard model, either it is true that $\Diamond\urcorner\alpha$, or it is true that $\Diamond S\alpha$, or it is true that it can be so that it is false that formulae ω and $\Omega\omega$ have identical logic values.

According to conditions VIII and IX, knowledge can be empirical even then, when it is neither fundamentally verifiable nor fundamentally falsifiable. This is already unhabitual (psychologically unexpected) as it goes far beyond traditional views of the classical empiricists. But this makes it possible simply to explain some still mysterious facts of history of cognition.

Truth conditions of/for formulae containing the expression $A\alpha$, are defined in this paper as follows:

X. For any formula α, if it is true that $A\alpha$, then it is true that $K\alpha$.

XI. For any formula α, if it is true that $A\alpha$, then it is true that α.

XII. For any formula α, if it is true that $A\alpha$, then it is true that $\Box\alpha$.

XIII. For any formula α, if it is true that $A\alpha$, then it is true that $\urcorner\Diamond S\alpha$.

XIV. For any formula α, if $A\alpha$, then for any formula ω, the formal-logic tantamount-ness $((\Omega\omega) \equiv \omega)$ is valid, or, differently speaking, ((formula $\Omega\omega$ has logic value "t"), if and only if (ω has logic value "t")), where symbol Ω stands for any element of the set of perfection modalities (this set is defined above).

XV. For any formula α, if $A\alpha$, then, for any formula ω, it is true that $(\Xi\omega \equiv \Omega\omega)$, where symbols Ξ and Ω stand for any elements of the set of perfection modalities.

The following statement is a concrete *particular* case (example) of the condition XIV. If $A\alpha$, then, for any term t_i, ((formula $T[t_i]$ has logic value "t") ≡ (formula $[t_i]$ has logic value "t")).

The following statements are concrete particular cases (examples) of the condition XV:

If $A\alpha$, then, for any formula ω, it is true that $(T\omega \equiv C\omega)$, where "T" stands for modality "it is true that …", and "C" stands for modality "it is consistent that …"

If $A\alpha$, then, for any formula ω, it is true that $(Ж\omega \equiv C\omega)$, where "Ж" stands for the existence modality, and "C" stands for the consistency modality.

Taking all the above-said into an account, it is easy to notice that, if the truth condition of/for$A\alpha$ is fulfilled, then the *modal collapse* takes place, namely: *all* the *de-dicto* modalities of perfection are logically equivalent to each other and can be eliminated from corresponding expressions without changing the logical values of these expressions, consequently, under the indicated extraordinary condition, the multimodal theory Ф+∃ "degenerates" (turns) into the consistent and complete classical calculus of propositions. It is generally believed that deducibility of the modal collapse is an existentially significant flaw of a (any) theory of modalities: the modality theory ceases to exist as such. Certainly, in general, this is really so. But, generally speaking, the modal collapse is not deducible in Ф+∃: in this theory, there is only a formal inference of the modal collapse (not in general but) from the very strong assumption (extraordinary hypothesis) that $A\alpha$. If $E\alpha$, then the modal collapse does not take place in Ф+∃. Hence, generally speaking, being taken as a whole, Ф+∃ is free of the modal-collapse problem.

I believe that, in future, the tendency to multimodality (or propensity to think of multiple kinds of modalities) exemplified in this article by formulae *Ga*, *Ta*, *Ca*, *Жa*, will head to a clearer recognizing (and more explicit and precise defining) of the essential ambiguity, polysemy and many-valued-ness of semantics of natural language. The present article is just the beginning of the indicated promising direction of scientific investigations.

3. Novel Scientific Results (A hitherto Unknown Formal Proof of Hilbert's Principle in the Newly Invented Logically Formalized Multimodal Axiomatic Epistemology-and-Ontology System Ф+∃)

Owving to the above-placed self-citations and self-references, the minimal set of exact definitions of basic notions of Ф+∃ which are *necessary and sufficient* for correct understanding and autonomous recheking the hitherto unpublished novel scientific results, now it is quite opportune to start generating (constructing) the above-promised significantly new formal deductive inferences. I mean the start of applying the hitherto never investigated axiomatic system Ф+∃ to the couple of conditional statements ST1 – ST2 and also to the pair of conditionals ((ST1*) – (ST2*)) formulated and discussed in the introduction, which conditional statements model Hilbert's ontology of mathematics. By means of the artificial language of Ф+∃, the conditional statements ST1* – ST2* (located in the introduction) are represented (modeled) by the following formulae (ST1+) – (ST2+), respectively.

ST1+: $(A\alpha \supset (C\omega \leftrightarrow Ж\omega))$.
ST2+: $(A\alpha \supset (C\omega \leftrightarrow T\omega))$.

The formulae (ST1+) and (ST2+) of Ф+∃ are representations (models) of the above-considered conditional statements (ST1*) and (ST2*), respectively. It is worth recalling here that, within a standard interpretation of Ф+∃, the symbol α in the formulae under considerarion is either a *proposition* (in particular, *proper mathematical* one), or a *theory* (in particular, *proper mathematical* one); the modal symbols C, T, Ж, A, respectively, denote the modalities "it is *Consistent* that…", "it is *True* that…", "what is described by…, *exists*". "it is *A-priori known* that …". Now let us move immediately to exact formulating and formal proving some schemes of theorems of Ф+∃. Concerning the mentioned theorem-schemes, the immediately following compllicated statement called "**Metatheorem-MT**" can be proved in Ф+∃.

Metatheorem-MT: The following theorem-schemes are formally provable in Ф+∃:
$(A\omega \supset (C\omega \leftrightarrow Ж\omega))$; $(A\omega \supset (C\omega \leftrightarrow T\omega))$; $(A\omega \supset (T\omega \leftrightarrow Ж\omega))$;
$(A\omega \supset (Y\omega \leftrightarrow Ж\omega))$; $(A\omega \supset (FЖ\omega \leftrightarrow PЖ\omega))$; $(A\omega \supset (PЖ\omega \leftrightarrow DЖ\omega))$;
$(A\omega \supset (FЖ\omega \leftrightarrow Ж\omega))$; $(A\omega \supset (PЖ\omega \leftrightarrow Ж\omega))$; $(A\omega \supset (\omega \leftrightarrow Ж\omega))$;
$(A\omega \supset (Ж\omega \leftrightarrow TЖ\omega))$; $(A\omega \supset (Ж\omega \leftrightarrow YЖ\omega))$; $(A\omega \supset (Ж\omega \leftrightarrow DЖ\omega))$;
$(A\omega \supset (Ж\omega \leftrightarrow \squareЖ\omega))$.

The below-placed finite succession (queue) of formula-schemes of Φ+∃ is a formal proof of the **Metatheorem-MT**.

1) $A\alpha \leftrightarrow (K\alpha \mathbin{\&} (\neg\Diamond\neg\alpha \mathbin{\&} \neg\Diamond S\alpha \mathbin{\&} \Box(\beta \leftrightarrow \Omega\beta))$: axiom-scheme AX-3.
2) $A\alpha \supset (K\alpha \mathbin{\&} (\neg\Diamond\neg\alpha \mathbin{\&} \neg\Diamond S\alpha \mathbin{\&} \Box(\beta \leftrightarrow \Omega\beta))$: from 1 by the rule of elimination of $\leftrightarrow$.
3) $A\alpha$: assumption.
4) $(K\alpha \mathbin{\&} (\neg\Diamond\neg\alpha \mathbin{\&} \neg\Diamond S\alpha \mathbin{\&} \Box(\beta \leftrightarrow \Omega\beta)$: from 2 and 3 by MP (*modus ponens*).
5) $\Box(\beta \leftrightarrow \Omega\beta)$: from 4 by the rule of elimination of &.
6) $(\beta \leftrightarrow \Omega\beta)$: from 3 and 5 by the unhabitual (newly formulated) rule[1] of elimination of $\Box$.
7) $A\alpha \mid\!- (\beta \leftrightarrow \Omega\beta)$: according to the train (sequence) 1—6.
8) $A\alpha \mid\!- (\beta \leftrightarrow \Xi\beta)$: from 7 by substitution of Ξ for Ω.
9) $A\alpha \mid\!- (\Xi\beta \leftrightarrow \beta)$: from 8 by the rule of commutativity of $\leftrightarrow$.
10) $A\alpha \mid\!- (\Xi\beta \leftrightarrow \Omega\beta)$: from 9 and 7 by the rule of transitivity of $\leftrightarrow$.
11) $A\alpha \mid\!- (\Omega\beta \leftrightarrow \beta)$: from 7 by the rule of commutativity of $\leftrightarrow$.
12) $\mid\!- (A\alpha \supset (\Omega\beta \leftrightarrow \beta))$: from 11 by the rule of introduction of $\supset$.
13) $\mid\!- (A\alpha \supset (\Xi\beta \leftrightarrow \Omega\beta))$: from 10 by the rule of introduction of $\supset$.
14) $\mid\!- (A\omega \supset (C\omega \leftrightarrow Ж\omega))$: from 13 by substitution of: (C for Ξ); ($Ж$ for Ω); (ω for α and β).
15) $\mid\!- (A\omega \supset (C\omega \leftrightarrow T\omega))$: from 13 by substitution of: (C for Ξ); (T for Ω); (ω for α and β).
16) $\mid\!- (A\omega \supset (T\omega \leftrightarrow Ж\omega))$: from 13 by substitution of: (T for Ξ); ($Ж$ for Ω); (ω for α and β).
17) $\mid\!- (A\omega \supset (Y\omega \leftrightarrow Ж\omega))$: from 13 by substitution of: (Y for Ξ); ($Ж$ for Ω); (ω for α and β).
18) $\mid\!- (A\omega \supset (FЖ\omega \leftrightarrow PЖ\omega))$: from 13 by substitution of: (F for Ξ); (P for Ω); (ω for α); ($Ж\omega$ for β).
19) $\mid\!- (A\omega \supset (PЖ\omega \leftrightarrow DЖ\omega))$: from 13 by substitution of: (P for Ξ); (D for Ω); (ω for α); ($Ж\omega$ for β).
20) $\mid\!- (A\omega \supset (FЖ\omega \leftrightarrow Ж\omega))$: from 12 by substitution of: (F for Ω); (ω for α); ($Ж\omega$ for β).
21) $\mid\!- (A\omega \supset (PЖ\omega \leftrightarrow Ж\omega))$: from 12 by substitution of: (P for Ω); (ω for α); ($Ж\omega$ for β).
22) $\mid\!- (A\omega \supset (KЖ\omega \leftrightarrow Ж\omega))$: from 12 by substitution of: (K for Ω); (ω for α); ($Ж\omega$ for β).
23) $\mid\!- (A\alpha \supset (\beta \leftrightarrow \Omega\beta))$: from 7 by the rule of introduction of $\supset$.
24) $\mid\!- (A\omega \supset (\omega \leftrightarrow Ж\omega))$: from 23 by substituting: ($Ж$ for Ω); (ω for α and β).
25) $\mid\!- (A\omega \supset (Ж\omega \leftrightarrow TЖ\omega))$: from 23 by substituting: (T for Ω); (ω for α); ($Ж\omega$ for β).
26) $\mid\!- (A\omega \supset (Ж\omega \leftrightarrow YЖ\omega))$: from 23 by substituting: (Y for Ω); (ω for α); ($Ж\omega$ for β).
27) $\mid\!- (A\omega \supset (Ж\omega \leftrightarrow DЖ\omega))$: from 23 by substituting: (D for Ω); (ω for α); ($Ж\omega$ for β).
28) $\mid\!- (A\omega \supset (Ж\omega \leftrightarrow \Box Ж\omega))$: from 23 by substituting: ($\Box$ for Ω); (ω for α); ($Ж\omega$ for β).

Here we are! Proving is ended.

The formal proof of the theorem-scheme ST1+: $(A\alpha \supset (C\omega \leftrightarrow Ж\omega))$ is nothing but the above-located succession (chain) of theorem-schemes 1—14. The formal proof of ST2+: $(A\alpha \supset (C\omega \leftrightarrow T\omega))$ is the above-provided sequence (queue) of theorem-schemes 1—15.

---

[1] The exact formulation of this inference-rule: $A\alpha, \Box\beta \mid\!— \beta$.

A corollary of the metatheorem-MT: from conjunction of 28) and $\neg(Ж\omega \leftrightarrow \Box Ж\omega)$, it follows logically (by *modus tollens*) that $\neg A\omega$. This corollary suites to history of philosophy wery well because, according to the *empiricism* [Юм, 1965, с. 517-518], in relation to exactly *empirical* knowledge of *facts*, $(Ж\omega \leftrightarrow \neg \Box Ж\omega)$.

4. Discussion of the Novel Results

If the above-formulated fundamental philosophical generalization of the mathematical-philosophy principle by Hilbert is adequately modeled by the conjunction $(A\omega \supset (C\omega \leftrightarrow Ж\omega))$ & $(A\omega \supset (C\omega \leftrightarrow T\omega))$, then, if the conjunction is false, then, according to Ф+∃, the knowledge is *not a priori but empirical* one. In other words, $(\neg(C\omega \leftrightarrow Ж\omega) \vee \neg(C\omega \leftrightarrow T\omega)) \supset \neg A\omega$. Thus, generally speaking, Kant's philosophy of mathematics (as exclusively *a priori* knowledge system) is to be rejected. (It is true not in general, but only partially.) Nonetheless, the falsity of the universal Kant's statement of a-priori-ness of proper mathematical knowledge does not undermine truthfulness of the conjunction (S1 & S2), which represents Hilbert's "fully-fleshed-out" principle of philosophy (epistemology-and-ontology) of proper mathematics. The conjunction (S1 & S2) is true due to the falsity of the antecedent of the classical (material) implications which are the conjuncts. Thus, being explicated and "fully-fleshed-out", Hilbert's principle is completely *deprived* of (or effectively *separated* from) its hidden Kant-epistemology foundation indirectly implied (presumed) in the above-cited Hilbert's letter to Frege. The above-formulated sentences S1* and S2*, significantly *generalizing* S1 and S2, respectively, are also true due to the falsity of their antecedent (statement of a-priori-ness of knowledge in general).

As in the classical propositional-logic algebra, the logic operation ⊃ is distributive in relation to the logic operation &, in principle, it is possible rationally to reduce the pair of conditionals modeling Hilbert's principle to the following one (let it be called S3): If Kant's philosophy-statement "mathematics is *a priori* knowledge" is true, then, in mathematics, ((consistency is equivalent to existence) and (consistency is equivalent to truth)). The significant generalization of S3 is represented by the statement S3*: in any knowledge sphere, if knowledge is *a priori*, then ((consistency is equivalent to existence) and (consistency is equivalent to truth)). In the formal multimodal axiomatic theory Ф+∃, the statement S3* is modeled by the formula-scheme S3+: $(A\omega \supset ((C\omega \leftrightarrow Ж\omega)$ & $(C\omega \leftrightarrow T\omega)))$.

In [Hintikka, 1962; 1974], the wonderful (somewhat surprising), enigmatic fact of Ancient Greeks' implicitly identifying (equalizing) in some relation: (a) knowledge and the object of knowledge; (b) knowledge and existence; (c) knowledge and truth had been noticed, recognized, and discussed systematically. Jaakko Hintikka assessed the mentioned curious identification fact as a somewhat strange (even paradoxical) event or even such a special tendency of intellectual history of Antiquity which had made up one of the most enigmatic (puzzling) aspects of Ancient Greek epistemology and ontology. In my opinion, the curious fact (enigmatic tendency) under discussion can be modeled in Ф+∃ by the formally provable formulae-schemes $(A\alpha \supset (K\alpha \leftrightarrow \alpha))$, $(A\alpha \supset (K\alpha \leftrightarrow Ж\alpha))$, and $(A\alpha \supset (K\alpha \leftrightarrow T\alpha))$, respectively.

The formula-scheme (Aα ⊃ (Tα ↔ Жα)), also formally provable in Ф+∃, deserves being discussed here as well. In some theology doctrines, it is proclaimed that "Truth" and "Existence" are Names of God [Pseudo-Dionysius Areopagite, 1980]; He is One. The Oneness of God implies the Oneness of "Truth" and "Existence", consequently, "Truth" and "Existence" are quite identical in some *enigmatic* meaning of the word "identity" (to be clarified and defined precisely). I think that one of quite rational options of clarifying and exact defining the enigmatic identity of "Truth" and "Existence" is the equivalence (Tα ↔ Жα) under the condition that Aα. In other words, (Aα ⊃ (Tα ↔ Жα)).

Finishing this article, I would like to discuss the following situation. At a conference on metamathematics, Bill has proved a meta-theorem of consistency of a formal theory X. During a coffee-break, while talking with Bill about the proof, Helen has made the following remark: "The formal theory X is an actually (Consistent and "Complete) Absurdity"! The remark has been accompanied by Helen's enigmatic (ambiguous) smile. This situation can be interpreted in qualitatively different (even opposite) ways. One of the possible interpretations implies taking seriously the following question. Is consistent and complete absurd (lie or nonsense), i.e. not-a-truth, possible?

Generally speaking, Kant's hypothetical (anticipated) answer to this question essentially depends of the special kind of knowledge implied in the question. Is the knowledge of proper mathematical theory X *empirical*, or is the proper mathematical theory X a formal system representing exclusively *a-priori* knowledge? Answers to these questions are essential for adequate solving the problem. If exactly *empirical* knowledge is meant, then Kant's anticipated answer to the question is to be positive: yes, it is possible, that the theory X is a consistent and complete absurd (lie or nonsense), i.e. not-a-truth. On the contrary, if exactly *a-priori* knowledge is meant, then Kant's anticipated answer to the question is to be negative: no, it is impossible, that the proper mathematical theory X is a consistent and complete absurd (lie or nonsense), i.e. not-a-truth. Trying to create an actually *universal* philosophical epistemology, Kant had to take into an account not only *a priori* but also empirical (*a posteriori*) knowledge. Nevertheless, he believed that all knowledge in formal logic and proper mathematics is *a priori* [Kant, 1994, 1996].

In accordance with Hilbert's deliberate accepting Kant's philosophy of mathematics, it is quite natural to anticipate the following Hilbert's hypothetic answer to the question under discussion: in proper pure mathematics it is not possible to encounter a consistent and complete absurd (lie or nonsense). According to the formalism ideal of self-sufficient mathematics, the situation, in which "The formal proper mathematical theory X is an actually (Consistent and "Complete) Absurdity" is impossible. This statement is modeled in Ф+∃ by the following theorem-schemes: Aω⊃ ¬◊(Cω & Yω & ¬Tω); ◊(Cω & Yω & ¬Ta) ⊃ ¬Aω.

Thus, adequate answering the question under discussion essentially depends on the relevant epistemic context (significant conditions), namely, on accepting or rejecting the assumption that knowledge is *a priori*. There is no logical contradiction between the hypothetical (anticipated) answers by Kant and by Hilbert as both believed that knowledge of pure formal logic and of proper pure mathematics is *a priori*. Hilbert was under strong influence by Kant's epistemology and ontology of mathematics [Lutskanov, 2010; Murawski, 2002; Zach, 2023]. However, according to Ф+∃, generally speaking, neither formal logic as a whole, nor proper pure mathematics as a whole, make up systems of pure *a priori* knowledge. In the light of Ф+∃, it is easy to see that both logic and mathematics (as wholes) are *empirical* knowledge systems. However, in the light of Ф+∃, it is also easy to see, that both *empirical* knowledge systems in question contain some existentially important *a priori* knowledge aspects (subsystems).

Список литературы / References

Айер, А. Дж. (2010). *Язык, истина и логика*. М.: «Канон+» РООИ «Реабилитация».

Ayer, A. J. (2010). *Language, Truth and Logic*. Moscow. (In Russ.)

Ершов, Ю. Л., Самохвалов, К. Ф. (2007). *Современная философия математики: недомогание и лечение*. Новосибирск: Параллель.

Ershov, Yu. L., Samokhvalov, K. F. (2007). *Contemporary Philosophy of Mathematics: Ailment and Treatment*. Novosibirsk. (In Russ.)

Лобовиков, В. О. (2014). Логический квадрат и гексагон эпистемических понятий. *Эпистемы: сборник научных статей. Вып. 9: Аспекты аналитической традиции*. Екатеринбург: Ажур. С. 57-68.

Lobovikov, V. O. (2014). Logical Square and Hexagon of Epistemic Notions. In *Epistemes. Collection of Scientific Papers. Iss. 9. Aspects of the Analytic Tradition*. Yekaterinburg. Pp. 57-68. (In Russ.)

Лобовиков, В. О. (2016). Уточнение формулировки и вариант решения проблемы эквивалентности деонтических, алетических и аксиологических модальностей с помощью аксиоматической системы философской эпистемологии. *Эпистемы: сборник научных статей. Вып. 11: Ценности и нормы в структуре рационального познания*. Екатеринбург: Макс-Инфо. С. 46-65.

Lobovikov, V. O. (2016). An Explication of Formulation and an Option of Solution of the Problem of Equivalence of of Deontic, Alethic and Axiological Modalities by Means of an Axiomatic System of Philosophical Epistemology. In *Epistemes. Collection of Scientific Papers. Iss. 11. Values and Norms in Structure of Rational Knowledge*. Yekaterinburg. Pp. 46-65. (In Russ.)

Лобовиков, В. О. (2023a). Искусственный интеллект и учение И. Канта о предписывании разумом законов природе (с точки зрения мультимодальной аксиоматической эпистемологии). *Трансцендентальный поворот в современной философии – 8: метафизика, эпистемология, когнитивистика и искусственный интеллект: сборник тезисов международной научной конференции. Москва, 20–22 апреля 2023 г.* Отв. ред. С. Л. Катречко, А. А. Шиян. М.: Изд-во: РГГУ. С. 108-111.

Lobovikov, V. O. (2023a). Artificial Intelligence and I. Kant's Doctrine of Reason's Prescribing Laws to Nature (from a Multimodal Axiomatic Epistemology Viewpoint). In Katrechko, S. L., Shiyan, A. A. (eds.). *Transcendental Turn in Contemporary Philosophy – 8. Metaphysics, Epistemology, Transcendental Cognitive Science and Artificial Intelligence. Proceedings of the International Workshop. Moscow, April 20–22, 2023.* Pp. 108-111. Moscow. (In Russ.)

Лобовиков, В. О. (2023б). Знания, онтологии, и мультимодальные аксиоматические теории (Дедуктивное доказательство непротиворечивости формальной теории $Ф^0$, синтезирующей формальную аксиологию, универсальную эпистемологию, и философскую онтологию). *Материалы IX международной конференции «Знания – Онтологии – Теории» (ЗОНТ-2023)*. Новосибирск, 2–6 октября 2023 г. Математический центр в Академгородке, Российская ассоциация искусственного интеллекта, Российская инженерная академия, Institute of Electrical and Electronics Engineers (IEEE). Новосибирск: Ин-т математики им. С. Л. Соболева СО РАН, НГУ. С. 202-211.

Lobovikov, V. O. (2023b). Knowledges, Ontologies, and Multimodal Axiomatic Theories (A Deductive Proof of Consistency of a Formal Theory $Ф^0$, Synthesizing Formal Axiology, Universal Epistemology, and Philosophical Ontology). In *Materials of IX International Conference "Knowledges – Ontologies – Theories". Novosibirsk, October 2–6, 2023.* Mathematics Center in Akademgorodok, Russian Association of Artificial Intelligence, Russian Engineering Academy, Institute of Electrical and Electronics Engineers (IEEE). Novosibirsk. Pp. 202-211. (In Russ.)

Поппер, К. (1983). *Логика и рост научного знания.* М.: Прогресс.
Popper, K. (1983). *Logic and Growth of Scientific Knowledge*. Moscow. (in Russ.)

Тарский, А. (1948). *Введение в логику и методологию дедуктивных наук.* М: Гос. изд-во иностр. лит-ры.
Tarski, A. (1948). *Introduction to Logic and to the Methodology of Deductive Sciences.* Moscow. (In Russ.)

Целищев, В. В. (1978). *Понятие объекта в модальной логике*. Новосибирск: Наука.
Tselishchev, V. V. (1978). *The Notion of Object in Modal Logic*. Novosibirsk. (In Russ.)

Целищев, В. В. (2003). *Онтология математики: объекты и структуры.* Новосибирск: Нонпарель.
Tselishchev, V. V. (2003). *Ontology of Mathematics: Objects and Structures.* Novosibirsk. (In Russ.).

Целищев, В. В. (2004а). Нормативный характер дедуктивных теорий и значение математических терминов. *Философия науки*. Т. 20. № 1. С. 71-82.

Tselishchev, V. V. (2004a). The normative character of deductive theories and the meaning of mathematical terms. *Philosophy of Science*. Vol. 20. No. 1. Pp. 71-82. (In Russ.)

Целищев, В. В. (2004б). *Нормативность дедуктивного дискурса: феноменология логических констант*. Новосибирск: Нонпарель.

Tselishchev, V. V. (2004b). *Normative-ness of Deductive Discourse: Phenomenology of Logic Constants*. Novosibirsk. (In Russ.)

Целищев, В. В. (2005). Непротиворечивость и полнота как нормы дедуктивного мышления в свете теорем Гёделя о неполноте арифметики. *Философия науки*. Т. 25. № 2. С. 34-52.

Tselishchev, V. V. (2005). Consistency and completeness as the standards of deductive thinking from the standpoint of Gödel's theorems about incompleteness of arithmetic. *Philosophy of Science*. Vol. 25. No. 2. Pp. 34-52. (In Russ.)

Целищев, В. В. (2010). *Логика существования*. М.: КРАСАНД.

Tselishchev, V. V. (2010). *Logic of Existence*. Moscow. (In Russ.)

Юм, Д. (1965). *Сочинения*. В 2 т. Т. 2. М.: Мысль.

Hume, D. (1965). *Works*. In 2 vols. Vol. 2. Moscow. (In Russ.)

Aristotle. (1994). The Works of Aristotle. Vol. 1. In Adler, M. J. (ed.). *Great Books of the Western World. Vol. 7*. Encyclopedia Britannica, Inc., Chicago; London.

Bader, R. M. (2021). Kantian Meta-Ontology. [Online]. *Academic library – free online college e-textbooks. The Routledge Handbook of Metametaphysics*. Routledge. Pp. 23-31. Available at: https://ebrary.net/255633/sociology/existence_modality#161 (Accessed: 10 December 2023).

Berto, F. (2012). *Existence as a Real Property*. Dordrecht. Springer.

Berto, F. and Priest, G. (2014). Modal Meinongianism and Characterization: Reply to Kroon, *Grazer. Philosophische Studien*. Vol. 90. Pp. 183-200.

Blanchette, P. A. (1996). Frege and Hilbert on Consistency. *Journal of Philosophy*. Vol. 93. No. 7. Pp. 317-336. DOI: 10.2307/2941124

Blanchette, P. A. (2007). Frege on Consistency and Conceptual Analysis. *Philosophia Mathematica*. Vol. 15. No. 3. Pp. 321-346. DOI:10.1093/philmat/nkm028

Blanchette, P. A. (2018). The Frege-Hilbert Controversy. [Online]. *The Stanford Encyclopedia of Philosophy Archive*. (Fall 2018 Edition). E. N. Zalta (ed.). Available at: https://plato.stanford.edu/archives/fall2018/entries/frege-hilbert/ (Accessed: 11 November 2023).

Blecher, I. (2012). *Kant and the Meaning of Existence: A Modal Account. PhD Dissertation*. Pittsburgh. University of Pittsburgh.

Bradie, M., Harms, W. (2020). Evolutionary Epistemology. [Online]. *The Stanford Encyclopedia of Philosophy Archive.* (Spring 2020 Edition). E. N. Zalta (ed.). Available at: https://plato.stanford.edu/archives/spr2020/entries/epistemology-evolutionary/ (Accessed: 11 November 2023).

Chihara, Ch. (1990). *Constructibility and Mathematical Existence*. Oxford. The University Press.

Corrigan, K., Harrington, L. M. (2023). Pseudo-Dionysius the Areopagite. [Online]. *The Stanford Encyclopedia of Philosophy Archive.* (Summer 2023 Edition). E. N. Zalta & U. Nodelman (eds.). Available at: https://plato.stanford.edu/archives/sum2023/entries/pseudo-dionysius-areopagite/ (Accessed: 11 November 2023).

Descartes, R. (1970). Meditations on First Philosophy, translated by E. Haldane and G. Ross. In *The Philosophical Works of Descartes.* Vol. 1. London. Cambridge University Press.

Doherty, F. T. (2017). Hilbert on Consistency as a Guide to Mathematical Reality. *Logique et Analyse*. No. 237. Pp. 107-128.

Fine, K. (1984). Critical Review of Parsons. *Nonexistent Objects. Philosophical Studies*. Vol. 45. Pp. 94-142.

Fine, K. (1985). *Reasoning With Arbitrary Objects*. Oxford. Blackwell.

Frege, G. (1971). *On the Foundations of Geometry and Formal Theories of Arithmetic.* Kluge, E.-H. W. (transl.). New Haven, CT. Yale University Press.

Frege, G. (1980). *Philosophical and Mathematical Correspondence.* Gabriel, G., Hermes, H., Kambartel, F., Thiel, Ch., Veraart, A., McGuinness, B., Kaal, H. (eds.) Oxford. Blackwell Publishers.

Frege, G. (1984). *Collected Papers on Mathematics, Logic and Philosophy*. McGuinness, B. F. (ed.). Oxford. Blackwell Publishers.

Haaparanta, L. (1985). Frege's Doctrine of Being. *Acta Philosophica Fennica.* Vol. 39. Helsinki. Societas Philosophica Fennica. 182 p.

Haaparanta, L. (1986). On Frege's Concept of Being. In Knuuttila, S., Hintikka, J. (eds.). *The Logic of Being.* Dordrecht. D. Reidel. Pp. 269-290.

Hartshorne, C. (1962). *The Logic of Perfection*. LaSalle, IL. Open Court.

Hartshorne, C. (1965). *Anselm's Discovery*. LaSalle, IL. Open Court.

Hilbert, D. (1990). *Foundations of Geometry*. Unger, L. (transl.) (2nd English ed.). La Salle, IL. Open Court Publishing Co.

Hilbert, D. (1996a). Axiomatic thought. In Ewald, W. B. (ed.). *From Kant to Hilbert: A Source Book in the Foundations of Mathematics*. Vol. 2. Oxford, New York. Oxford University Press Inc. Pp. 1105-1115.

Hilbert, D. (1996b). The new grounding of mathematics: First report. In Ewald, W. B. (ed.). *From Kant to Hilbert: A Source Book in the Foundations of Mathematics*. Vol. 2. Oxford, New York. Oxford University Press Inc. Pp. 1115-1134.

Hilbert, D. (1996c). The logical foundations of mathematics. In Ewald, W. B. (Ed.). *From Kant to Hilbert: A Source Book in the Foundations of Mathematics*. Vol. 2. Oxford, New York. Oxford University Press Inc. Pp. 1134-1147.

Hintikka, J. (1962). *Knowledge and Belief: An Introduction to the Logic of the Two Notions*. Ithaca, NY. Cornell University Press.

Hintikka, J. (1969). *Models for Modalities. Selected Essays*. Dordrecht. D. Reidel Publishing Co.

Hintikka, J. (1974). *Knowledge and the Known: Historical Perspectives in Epistemology*. Dordrecht. D. Reidel Publishing Co.

Hintikka, J. (1984). Are There Nonexistent Objects? Why Not? But Where Are They? *Synthese*. Vol. 60. Pp. 451-458.

Hintikka, J. (1986). Kant, Existence, Predication and the Ontological Argument. In Knuuttila, S., Hintikka, J. (eds.). *The Logic of Being*. Dordrecht. D. Reidel. Pp. 249-268.

Jacquette, D. (1996). On Defoliating Meinong's Jungle. *Axiomathes*. No. 1-2. Pp. 17-42.

Jacquette, D. (1997). *Meinongian Logic. The Semantics of Existence and Nonexistence*. Berlin. New York. Walter de Gruyter.

Jacquette, D. (2015). Alexius Meinong. The Shepherd of Non-Being. Vol. 360. [Online]. *Springer Link. Synthese Library*. Available at: https://link.springer.com/book/10.1007/978-3-319-18075-5 (Accessed: 11 November 2023).

Janowitz, N. (1991). Theories of Divine Names in Origen and Pseudo-Dionysius. *History of Religions*. Vol. 30, Pp. 359-372.

Kant, I. (1994). The Critique of Pure Reason. Fundamental Principles of the Metaphysics of Morals. The Critique of Practical Reason. Preface and Introduction to the Metaphysical Elements of Ethics. General Introduction to the Metaphysics of Morals. The Science of Right. The Critique of Judgement. In Adler, M. J. (ed.). *Great Books of the Western World. V. 39. Kant*. Chicago. London. Encyclopedia Britannica, Inc.

Kant, I. (1996). *Prolegomena to any future metaphysics: in focus*. London. New York. Routledge.

Kneale, W. (1962). Modality de Dicto and de Re. In Nagel, E., Suppes, P. and Tarski, A. (eds.). *Logic, Methodology and Philosophy of Science*. Stanford. Stanford University Press. Pp. 622-633.

Leibniz, G. W. (1969). *Philosophical Papers and Letters*. Dordrecht, Netherlands. D. Reidel.

Leibniz, G. W. (1981). *New Essays on Human Understanding*. Cambridge, UK. Cambridge University Press.

Lobovikov, V. O. (2020). Knowledge Logic and Algebra of Formal Axiology: A Formal Axiomatic Epistemology Theory Sigma Used for Precise Defining the Exotic Condition under which Hume-and-Moore Doctrine of Logically Unbridgeable Gap between Statements of Being and Statements of Value is Falsified. *Antinomies*. Vol. 20. No. 4. Pp. 7-23. DOI: 10.24411/2686-7206-2020-10401

Lobovikov, V. O. (2021). A Logically Formalized Axiomatic Epistemology System Σ + C and Philosophical Grounding Mathematics as a Self-Sufficing System. *Mathematics*. Vol. 9. No. 16. Pp. 1859. DOI: 10.3390/math9161859

Lobovikov, V. O. (2022). Axiomatizing Philosophical Epistemology, a Formal Theory "Sigma + 2C", and Philosophical Foundations of Mathematics. *Аналитическая философия: траектории истории и векторы развития: сборник научных трудов Международной научной конференции, посвященной 80-летию научного руководителя Института философии и права СО РАН В. В. Целищева, Новосибирск, 25–26 февраля 2022 г.* А. В. Хлебалин (ред.). Новосибирск: Офсет ТМ. С. 43-52.

Lobovikov, V. O. (2022). Axiomatizing Philosophical Epistemology, a Formal Theory "Sigma + 2C", and Philosophical Foundations of Mathematics. In Hlebalin, A. V. (ed.). *Analytical Philosophy: History Trajectories and Development Vectors. Proceedings of the International Scientific Conference Devoted to 80th Anniversary of V. V. Tselischev – Scientific Leader of the Institute of Philosophy and Law of Siberian Branch of Russian Academy of Sciences. Novosibirsk, February 25–26, 2022.* Novosibirsk. Pp. 43-52.

Lobovikov, V. O. (2023). Combining Universal Epistemology with Formal Axiology in a Multimodal Formal Axiomatic Theory “Sigma+2C”, and Philosophical Foundations of Mathematics. *Respublica Literaria*. Vol. 4. No. 4. Pp. 88-113. DOI: 10.47850/RL.2023.4.4.88-113

Logan, I. (2009). *Reading Anselm’s Proslogion: The History of Anselm’s Argument and its Significance Today*. Aldershot. Ashgate.

Lutskanov, R. (2010). Hilbert’s Program: the Transcendental Roots of Mathematical Knowledge. *Balcan Journal of Philosophy*. Vol. 2. Iss. 2. Pp. 121-126.

Mancosu, P. (ed.). (1998). *From Brouwer to Hilbert. The Debate on the Foundations of Mathematics in the 1920s*. Oxford. Oxford University Press.

Marek, J. (2022). Alexius Meinong. [Online]. *The Stanford Encyclopedia of Philosophy Archive.* (Fall 2022 Edition). E. N. Zalta & U. Nodelman (eds.). Available at: https://plato.stanford.edu/archives/fall2022/entries/meinong/ (Accessed: 11 November 2023).

Meinong, A. (1960). On Object Theory. In Chisholm, R. (ed.). *Realism and the Background of Phenomenology*. Glencoe. The Free Press.

Montague, R. (1960). Logical Necessity, Physical Necessity, Ethics, and Quantifiers. *Inquiry: An Interdisciplinary Journal of Philosophy.* Vol. 3. Pp. 259-269. DOI: 10.1080/00201746008601312

Moschovakis, J. (2007). Intuitionistic Logic. [Online]. *The Stanford Encyclopedia of Philosophy Archive.* (Spring 2007 Edition). E. N. Zalta (ed.). Available at: https://plato.stanford.edu/archives/spr2007/entries/logic-intuitionistic/ (Accessed: 11 November 2023).

Murawski, R. (2002). Leibniz's and Kant's Philosophical Ideas and the Development of Hilbert's Programme. *Logique et Analyse*. Vol. 45. No. 179/180. Pp. 421-437.

Nelson, M. (2022). Existence. [Online]. *The Stanford Encyclopedia of Philosophy Archive.* (Winter 2022 Edition). E. N. Zalta & U. Nodelman (eds.). Available at: https://plato.stanford.edu/archives/win2022/entries/existence/ (Accessed: 11 November 2023).

Neurath, O. (1982). Pseudorationalism of Falsification. In *Philosophical Papers, 1913-1946. Vienna Circle collection. Vol. 16*. Dordrecht, Holland. D. Reidel. Pp. 121-131.

Nolan, L. (2021). Descartes’ Ontological Argument. [Online]. *The Stanford Encyclopedia of Philosophy Archive.* (Spring 2021 Edition). Edward N. Zalta (ed.). Available at: https://plato.stanford.edu/archives/spr2021/entries/descartes-ontological/ (Accessed: 4 January 2024).

Oppenheimer, P., Zalta, E. (1991). On the Logic of the Ontological Argument. In Tomberlin J. (ed.). *Philosophical Perspectives. 5: The Philosophy of Religion*. Atascadero. Ridgeview. Pp. 509-529.

Oppy, G. (2023). Ontological Arguments. [Online]. *The Stanford Encyclopedia of Philosophy Archive*. (Fall 2023 Edition). E. N. Zalta & U. Nodelman (eds.). Available at: https://plato.stanford.edu/archives/spr2023/entries/ontological-arguments/ (Accessed: 4 January 2024).

Parsons, T. (1980). *Nonexistent Objects*. New Haven. Yale University Press.

Parsons, T. (1982). Are There Nonexistent Objects? *American Philosophical Quarterly*. Vol. 19. Pp. 365-371.

Perszyk, K. J. (1993). *Nonexistent Objects. Meinong and Contemporary Philosophy*. Dordrecht. Kluwer.

Plato. (1994). The Dialogues of Plato. In Adler, M. J. (ed.). *Great Books of the Western World. Vol. 6: Plato*. Chicago. London. Encyclopedia Britannica, Inc. Pp. 1-799.

Poincaré, H. (1912). The Latest Efforts of the Logisticians. *The Monist*. Vol. 22. No. 4. Pp. 524-539. [Online]. *JSTOR*. Available at: https://www.jstor.org/stable/27900395 (Accessed: 02 February 2024).

Priest, G. (2005). *Towards Non-Being. The Logic and Metaphysics of Intentionality*. Oxford. Clarendon.

Prior, A. N. (1961). Non-Entities. *Philosophical Review*. Vol. 70. Pp. 120-132.

Pseudo-Dionysius Areopagite. (1980). *The Divine Names and Mystical Theology*. Jones, J. (transl.). Milwaukee, Wisconsin. Marquette University Press.

Resher, N. (1959). On Logic of Existence and Denotation. *Philosophical Review*. Vol. 68. Pp. 157-180.

Quine, W. (1969). *Ontological Relativity and Other Essays*. New York, NY. Columbia University Press.

Quine, W. (1980). *From a Logical Point of View. Nine Logico-Philosophical Essays*. Cambridge, MA. Harvard University Press.

Reicher, M. (2022). Nonexistent Objects. [Online]. *The Stanford Encyclopedia of Philosophy Archive*. (Winter 2022 Edition). E. N. Zalta & U. Nodelman (eds.). Available at: https://plato.stanford.edu/archives/win2022/entries/nonexistent-objects/ (Accessed: 4 January 2024).

Rendsvig, R., Symons, J., Wang, Y. (2023). Epistemic Logic. [Online]. *The Stanford Encyclopedia of Philosophy Archive*. (Winter 2023 Edition). E. N. Zalta & U. Nodelman (eds.). Available at: https://plato.stanford.edu/archives/win2023/entries/logic-epistemic/ (Accessed: 4 January 2024).

Rosefeldt, T. (2020). Kant's Logic of Existence. *Journal of the History of Philosophy*. Vol. 58. No. 3. Pp. 521-548.

Routley, R. (1980). *Exploring Meinong's Jungle and Beyond*. Canberra. Australian National University.

Russell, B. (1905). On Denoting. *Mind*. Vol. 14. No. 56. Pp. 479-493.

Russell, B. (1941). *An Inquiry into Meaning and Truth*. New York. W. W. Norton & Company.

Russell, B. (1992). Theory of Knowledge: the 1913 Manuscript. In Eames, E. R. and Blackwell, K. (eds.). *Collected Papers of Bertrand Russell*. London and New York. Routledge.

Shanker, S. G. (1988). Wittgenstein's remarks on the significance of Gödel's theorem. In Shanker, S. G. (ed.) *Gödel's theorem in focus*. London: Routledge, pp. 155-257.

Smith, J. F. (1985). The Russell-Meinong Debate. *Philosophy and Phenomenological Research*. Vol. 45. No. 3. Pp. 305-350.

Thomason, R. H. (ed.). (1974). *Formal Philosophy. Selected Papers of Richard Montague*. New Haven. Yale University Press.

Williams, T. (2023). Anselm of Canterbury. [Online]. *The Stanford Encyclopedia of Philosophy Archive*. (Fall 2023 Edition). E. N. Zalta & U. Nodelman (eds.). Available at: https://plato.stanford.edu/archives/fall2023/entries/anselm/. (Accessed: 4 January 2024).

Williamson, T. (2013). *Modal Logic as Metaphysics*. Oxford. Oxford University Press.

Wolenski, J. (2016). *Truth as Modality*. Rzeszow, Poland. University of Information, Technology and Management. [Online]. Available at: http://ojs.philosophy.spbu.ru/index.php/lphs/article/viewFile/370/377 (Accessed: 4 January 2024).

Wright, G. H. (1951). *An Essay in Modal Logic* (Studies in Logic and the Foundations of Mathematics). Amsterdam. North-Holland Publishing Company.

Wright, G. H. (1996). Truth-Logics. In *Georg Henrik von Wright. Six Essays in Philosophical Logic. Acta Philosophica Fennica*. Vol. 60. Pp. 71-91. Helsinki. Societas Philosophica Fennica.

Zach, R. (2023). Hilbert's Program. [Online]. *The Stanford Encyclopedia of Philosophy Archive*. (Winter 2023 Edition). E. N. Zalta & U. Nodelman (eds.). Available at: https://plato.stanford.edu/archives/win2023/entries/hilbert-program/ (Accessed: 4 January 2024).

Zalta, E. N. (1983). *Abstract Objects: An Introduction to Axiomatic Metaphysics*. Dordrecht. D. Reidel.

Information about the author / Информация об авторе

Lobovikov Vladimir – Doctor of Philosophical Sciences, Professor, Principal Researcher, Institute of Philosophy and Law of the Ural Branch of the Russian Academy of Sciences, Ekaterinburg, Sofya Kovalevskaya Str., 16, e-mail: vlobovikov@mail.ru, https://orcid.org/0000-0001-8760-0452.

Статья поступила в редакцию: 08.01.2024
После доработки: 04.03.2024
Принята к публикации: 14.03.2024

Лобовиков Владимир Олегович – доктор философских наук, профессор, главный научный сотрудник Института философии и права Уральского отделения Российской академии наук, г. Екатеринбург, ул. Софьи Ковалевской, 16, e-mail: vlobovikov@mail.ru, https://orcid.org/0000-0001-8760-0452.

The paper was submitted: 08.01.2024
Received after reworking: 04.03.2024
Accepted for publication: 14.03.2024