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A Logically Formalized Axiomatic Epistemology System Σ + C and Philosophical Grounding Mathematics as a Self-Sufficing System

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Abstract: The subject matter of this research is Kant's apriorism underlying Hilbert's formalism in the philosophical grounding of mathematics as a self-sufficing system. The research aim is the invention of such a logically formalized axiomatic epistemology system, in which it is possible to construct formal deductive inferences of formulae—modeling the formalism ideal of Hilbert—from the assumption of Kant's apriorism in relation to mathematical knowledge. The research method is hypothetical–deductive (axiomatic). The research results and their scientific novelty are based on a logically formalized axiomatic system of epistemology called Σ + C, constructed here for the first time. In comparison with the already published formal epistemology systems Ξ and Σ, some of the axiom schemes here are generalized in Σ + C, and a new symbol is included in the object-language alphabet of Σ + C, namely, the symbol representing the perfection modality, C: "it is consistent that ... ". The meaning of this modality is defined by the system of axiom schemes of Σ + C. A deductive proof of the consistency of Σ + C is submitted. For the first time, by means of Σ + C, it is deductively demonstrated that, from the conjunction of Σ + C and either the first or second version of Gödel's theorem of incompleteness of a formal arithmetic system, the formal arithmetic investigated by Gödel is a representation of an empirical knowledge system. Thus, Kant's view of mathematics as a self-sufficient, pure, a priori knowledge system is falsified.

Keywords: logically formalized axiomatic system of epistemology; a priori knowledge; empirical knowledge; Kant's apriorism; Hilbert's formalism; Gödel's incompleteness theorem; two-valued algebraic system of formal axiology

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1. Introduction

> "Before all, be it observed, that proper mathematical propositions are always judgements a priori, and not empirical, because they carry along with them the conception of necessity, which cannot be given by experience".
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> I. Kant [1] (p. 18).

> "The denial of deflationism leads us to a difference in the concepts of truth and proof. For the strict formalist, truth and proof are of course equivalent concepts: a sentence of mathematics is true if and only if it is provable, and as such truth is a deflationary concept. ... the formalist believes that truth and proof are the same concept by definition".
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> M. Pantsar [2] (p. 20).

There are infinitely many different modal logics. The quantity of possible combinations of different kinds of modality is immense. Even within the scope of the modal logic of knowledge, we need a set of significantly different modalities called "knowledge", various combinations of which constitute different multimodal epistemic logics. Given that, in the broadly accepted definitions of "knowledge", the words "true" and "provable" are exploited necessarily, the modal logic treating truth as modality [3,4] and the modal logic treating provability as modality [5–10] are indispensable to epistemology. In content philosophy, the word "knowledge" is naturally connected with many other modal terms (alethic, deontic, axiological, etc.); consequently, while inventing and elaborating a hypothetical multimodal formal axiomatic system of universal philosophical epistemology, one has to utilize not only proper epistemic modalities, but also many other concepts of modal metaphysics. This is the aim of the present article, namely, the invention (construct) of a novel, logically formalized, axiomatic system for a multimodal philosophy of knowledge. However, the concrete theme and the goal of the paper necessitate a restriction to the set of different kinds of modalities involved in the discourse.

The list of various kinds of modality to be addressed in this article is determined by the subject matter and the target of the research. At the present moment, we are equipped with the axiomatic epistemology system Σ, which was previously addressed in [11]. Is Σ sufficient for realizing the goal of the paper? If not, then which new modalities are to be added to Σ to make it an effective instrument of/for realizing the goal? To answer these questions adequately, we begin with a systematic explication of the subject matter of the research, its aim, and the means of/for realizing this aim.

In this article, the logically formalized multimodal axiomatic epistemology system Σ (or an option of its mutation) is used for the logical analysis of a system of philosophical foundations for mathematics, which (system) is made up of the following set of statements, ST1–ST5:

ST1—Proper mathematical knowledge of ω is a priori. See, for instance, Kant [1] (pp. 16, 18);

ST2—The truth of ω and the provability of ω are logically equivalent in the rationalistic optimism ideal created by Leibniz [12–14] and Hilbert [15–20]. For the rationalistic optimism ideal and Gödel's philosophy, see also [21]. Incidentally, here, it is relevant to note that there is a nontrivial formal-axiological equivalence of "true" and "provable" [22], but the almost unknown "formal-axiological equivalence" and the well-known "formal-logical equivalence" are not identical;

ST3—The consistency of proper mathematical statement or theory ω and the provability of consistency of ω are logically equivalent in Hilbert's ideal of self-sufficient (self-dependent) mathematics;

ST4—The truth of ω and the consistency of ω are logically equivalent (in the ideal);

ST5—The truth of ω is logically equivalent to ω (in the ideal).

D. Hilbert was not alone; his rationalistic optimism ideal (norm) of mathematical activity was also attractive for Tarski and many other prominent mathematicians. Even while being aware of Gödel's theorems of incompleteness, Tarski believed and wrote that it is good (desirable) for a mathematician to prove that ST2 is true in relation to a concrete mathematical statement or theory ω, if this proof is possible [23] (pp. 185–189). While being aware of Gödel's theorems of incompleteness and taking them into account, Tselishchev wrote (in perfect accordance with Tarski) that proving consistency and completeness is a prescribed (obligatory) norm (duty) for a mathematician, if such proof is possible [24–26]. Here, the famous bimodal Kantian principle "obligation (duty) implies possibility" ($Op \supset \Diamond p$) applies. As, according to the theorems of Gödel, proving the completeness of a formal arithmetic system (under the condition of its consistency) is impossible, there is no violation of the norm (the relevant obligation is abolished by modus tollens).

If Hilbert's formalism ideal and program of/for the philosophical grounding of mathematics were to be fulfilled (i.e., if the ideal created by him was realized), then the system of mathematical knowledge (as a whole) would be self-sufficient (self-dependent). Unfortunately, there is today a widespread opinion (a statistical norm of thinking and affirming) that Hilbert's ideal, and the formalism program targeted at realizing this ideal, were totally annihilated by Gödel's theorems of incompleteness. However, this widespread opinion is not able to explain the philosophical foundation of/for Hilbert's creation of the ideal and the formalism program in question. Those talking of Gödel's termination of Hilbert's formalism program do not recognize the possibility of the existence of a non-empty domain, in which context Hilbert's ideal and the formalism program targeted at realizing this ideal are perfectly adequate even today (and will remain so forever). As such, the significance of Gödel's famous work is reduced to the narrow limiting of the abovementioned non-empty domain, i.e., to the significance of establishing precise borders of/for that domain. The present paper aims at recognizing and explicating the reason of/for Hilbert's creation of the formalism program, and at giving an exact definition of the realm of the program's soundness, which is missed by the aforementioned papers.

By scrutinizing the above statements ST1–ST5, it is easy to extract a set of modalities that are indispensable for formulating ST1–ST5, as follows: "*knows* that ... "; "*a-priori knows* that ... "; "*empirically knows* that ... "; "*it is true* that ... "; "it is *provable* in the consistent theory that ... ", "it is *consistent* that ... ". The first five modalities are taken into an account by Σ, whereas the last is not. Therefore, to successfully address the research goal, it is necessary to modify Σ by adding the novel modality "*Consistency*" to it. Let the symbol $C\omega$ stand for "it is *consistent* that ω". Given that the novel modality "C" (Consistency) is now added to Σ, let the symbol "Σ + C" be the name of/for the result of this addition to Σ. The general concept of this article is as such addressed in its basic form. Now let us move to the next paragraph, giving a precise definition of the multimodal formal axiomatic system Σ + C to be exploited in this paper.

2. Materials and Methods

We applied the abstract concepts and methods of discrete mathematics (especially algebra) and symbolic logic (especially modal and multimodal) to the materials of universal philosophical ontology, epistemology, axiology, history of philosophy, and the philosophical foundations of mathematics. The traditional materials and methods of the history of philosophy have been complemented by the axiomatic (hypothetic–deductive) method of logical reasoning. Not only has content analysis been exploited, but so too has formal theory investigation (formal inference construction).

3. Results

3.1. A Hitherto Never Published Logically Formalized Multimodal Axiomatic Epistemology and Axiology System Σ + C

As a result of adding the modality C ("It is *consistent* that ... ") to the set of *perfection* modalities of the multimodal system Σ, and significantly *generalizing* the axiom scheme AX-5 of Σ, a new system (named "Σ + C") has come into being. The axiomatic system Σ + C is a result of developing the formal axiomatic epistemological theory Ξ [27] and the formal axiomatic epistemological and axiological theory Σ [11]. In order to not repeat key sentences and phrases from my own previously published work, I have not given precise definitions of the notions "*alphabet* of object-language of Σ + C", "*term* of Σ + C", "*formula* of Σ + C", and a few others. Definitions of these notions in the context of Σ + C are similar to corresponding definitions in the context of Σ. The reader can find such definitions in my already published (open access) work [28,29].

The proper logic axioms and inference rules of Σ and Σ + C are the same as those used in the classical logic of propositions. Thus, the proper logical foundations of Σ and Σ + C are identical, but the logically formalized systems constructed on these foundations are different. Although, at first glance, the definitions of Σ and Σ + C seem identical, strictly speaking, they are not. The object-languages of Σ and Σ + C have different alphabets, and the theories contain different sets of expressions, different sets of formulae, different sets of axioms, and different sets of theorems.

The modality symbols exploited in the present article are as follows. K, A, E, S, T, F, P, and D, respectively, stand for the modalities "agent *Knows* that ... "; "agent *A-priori knows* that ... "; "agent *Empirically* (*a-posteriori) knows* that ... "; "under some conditions in some space-and-time a person (immediately or by means of some tools) *Sensually perceives* (has *Sensual verification*) that ... "; "it is *True* that ... "; "person has *Faith* (or believes) that ... ";

"it is *Provable* in the consistent theory that . . . "; and "there is *an algorithm* (a machine could be constructed) *for Deciding* that . . . ".

C, G, W, O, B, U, J, respectively, stand for the modalities "it is *Consistent* that . . . "; "it is *(morally) Good* that . . . "; "it is *(morally) Wicked* that . . . "'; "it is *Obligatory* that . . . "; "it is *Beautiful* that . . . "; "it is *Useful* that . . . "; and "it *is Joyful, pleasant* that . . . ".

In this paragraph of the article, the syntactical meanings of the modality symbols are defined precisely (although not manifestly) by the following system of schemes of proper axioms of multimodal philosophy theory $\Sigma + C$.

Axiom scheme AX-1: $A\alpha \supset (\Box\beta \supset \beta)$.
Axiom scheme AX-2: $A\alpha \supset (\Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta))$.
Axiom scheme AX-3: $A\alpha \leftrightarrow (K\alpha \,\&\, (\neg\Diamond\neg\alpha \,\&\neg\Diamond\, S\alpha \,\&\, \Box(\beta \leftrightarrow \Omega\beta)))$.
Axiom scheme AX-4: $E\alpha \leftrightarrow (K\alpha \,\&\, (\Diamond\neg\alpha \vee \Diamond S\alpha \vee \neg\Box(\beta \leftrightarrow \Omega\beta)))$.
Axiom scheme AX-5: $\Omega\alpha \supset \Diamond\alpha$.
Axiom scheme AX-6: $(\Box\beta \,\&\, \Box\Box\beta) \supset \beta$.
Axiom scheme AX-7: $(t_i = + = t_k) \leftrightarrow (G[t_i] \leftrightarrow G[t_k])$.
Axiom scheme AX-8: $(t_i = + = g) \supset \Box G[t_i]$.
Axiom scheme AX-9: $(t_i = + = b) \supset \Box W[t_i]$.
Axiom scheme AX-10: $(G\alpha \supset \neg W\alpha)$.
Axiom scheme AX-11: $(W\alpha \supset \neg G\alpha)$.

Definition scheme DF-1: $\Diamond\gamma$ is a *name* of/for $\neg\Box\neg\gamma$ (where γ is a formula of $\Sigma + C$).

In AX-3 and AX-4, the symbol Ω (belonging to the meta-language) only represents a (any) "perfection-modality". Not all of the above-mentioned modalities are perfection-modalities. The set Δ of "perfections" (perfection-modalities) is {K, D, F, C, P, J, T, B, G, U, O, $\Box$}.

Evidently, Δ is a subset of the set of all modalities under consideration in this article. Incorporating C into the set Δ of *perfection*-modalities is quite natural as "consistency" is an important *perfection* of a theoretical system [23] (pp. 185, 186). As a rule, *de dicto* modalities are attached to a *dictum*. Usually, the word "*dictum*" is translated from Latin as "*proposition* (or sentence)", but, in principle, it is possible to generalize its meaning in such a manner that a theoretical (deductive) system would be a *dictum* as well.

A justification of AX-10 and AX-11 can be found in the book [30] devoted to the formal logic of evaluations. However, the almost unknown (unhabitual) axiom schemes AX-7, AX-8, and AX-9 represent not the formal logic, but a formal axiology (universal theory of value forms). The notion "formal logic" is not logically equivalent to the notion "formal axiology"; consequently, "formal logic inconsistency" and "formal axiological inconsistency" are not synonyms. The significant logical difference between the notions "*formal-axiological* contradiction" and "formal logical contradiction" explains the non-intuitive possibility of a deductive proof of the *formal-axiological inconsistency* of formal arithmetic theory [31,32].

3.2. Defining Semantics of/for the Multimodal Formal Theory $\Sigma + C$

In the above section, the definition of $\Sigma + C$ has been deliberately deprived of its philosophical content (owing to the relevant abstraction). The axiomatic theory $\Sigma + C$ is a *multimodal* one, but thus far, the actual contents of modalities have not been addressed sufficiently, and the theory of $\Sigma + C$ has been considered to be a formal one. In this section, we attempt to relax the formality of $\Sigma + C$ and address the actual meaning of the modalities under consideration in the multimodal theory $\Sigma + C$. On the syntactical level of the artificial language of $\Sigma + C$, its modal symbols are defined by the above axiom schemes (AX1–AX11).

In the present article, it is presumed that the semantic meanings of the proper logical symbols of the artificial language of classical symbolic logic are already well defined. As the proper logic symbols are well known, there is no need to define them here. On the contrary, the unusual symbols of the artificial language of $\Sigma + C$ require systematic definition.

Defining the semantic meanings involves defining the interpretation function. To define the interpretation function, one has to define (1) the set that plays the role of "domain (or field) of interpretation" (let the interpretation domain be denoted by the letter M) and

(2) a "valuator (evaluator)", *V*. In a standard interpretation of Σ + C, M is a set in which every element has: (1) one and only one *axiological value* from the set {good, bad}; (2) one and only one *ontological value* from the set {exists, not-exists}.

The *axiological variables* (z, x, y, z_i, x_k, y_m) take their values from the set M.

The *axiological constants* "b" and "g" mean "bad" and "good", respectively.

Valuating an element of M using a concrete (fixed) interpreter V involves ascribing an *axiological value* (either good or bad) to that element. The interpreter V may be either collective or individual. A change in V can definitely change some of the relative evaluations, but it cannot change the laws of the two-valued algebra of formal axiology, which are absolute rather than relative evaluations; that is, they are such and only such *constant valuation-functions* that have the value g (good) under any possible combination of values of their axiological variables. Although V is a variable that takes its value from the set of all possible interpreters, a perfectly defined interpretation of Σ + C necessarily implies that the value of V is fixed. A change in V necessarily implies a change in interpretation.

In the present article, "e" and "n" represent " ... exists" and " ... not-exists", respectively. The signs "e" and "n" are "*ontological constants*". In a standard interpretation of Σ + C, one and only one element of the set {{g, e}, {g, n}, {b, e}, {b, n}} corresponds to every element of M. The signs "e" and "n" belong to the meta-language. By definition, "e" and "n" do not belong to the alphabet of the object-language of Σ + C. Nevertheless, "e" and "n" are *indirectly* represented in the object-language of Σ + C by means of *square-bracketing*: "t_i exists" is represented by $[t_i]$; "t_i does not exist" is represented by $\neg[t_i]$. This means that square-bracketing is intrinsic to the formal-axiological and ontological semantics of Σ + C.

N-placed terms of Σ + C are interpreted as *n*-placed evaluation-functions, defined on the set M. "One-placed evaluation-functions" are laid out in Table 1. (Here, the upper index 1 immediately after a capital letter means that this letter represents a one-placed evaluation-function).

Table 1. Definition of the functions determined by one evaluation argument.

x	$B_1{}^1x$	$N_1{}^1x$	$C_1{}^1x$	$I_1{}^1x$	$Z_1{}^1x$	$S_1{}^1x$	$U_1{}^1x$	$A_1{}^1x$	$G_1{}^1x$	$P_1{}^1x$	$H_1{}^1x$	$R_1{}^1x$
g	g	b	g	b	b	b	b	g	g	g	b	b
b	b	g	b	g	b	b	b	g	g	b	g	g

In Table 1, the one-placed term $B_1{}^1x$ is interpreted as the one-placed evaluation-function "being (existence) of (what, whom) x"; $N_1{}^1x$ is the evaluation-function "non-being (nonexistence) of (what, whom) x". $C_1{}^1x$—"consistency of (what, whom) x"; $I_1{}^1x$—"inconsistency of (what, whom) x"; $Z_1{}^1x$—"formal-axiological inconsistency (or absolute inconsistency) of (what, whom) x"; $S_1{}^1x$—"x's self-contradiction"; $U_1{}^1x$—"absolute non-being of (what, whom) x"; $A_1{}^1x$—"absolute being of (what, whom) x"; $G_1{}^1x$—"absolute goodness of (what, whom) x", or "absolute good (what, who) x"; $P_1{}^1x$—"positive evaluation of (what, whom) x"; $H_1{}^1x$—"negative evaluation of (what, whom) x"; $R_1{}^1x$—"resistance to (what, whom) x". Concerning additional examples of one-placed evaluation-functions, see [11,27,33].

*Two-placed evaluation-function*s are shown in Table 2. (Here, the upper index 2 immediately after a capital letter means that this letter represents a two-placed function).

Table 2. Definition of the evaluation-functions determined by two arguments.

x	y	K^2xy	S^2xy	X^2xy	T^2xy	Z^2xy	P^2xy	C^2xy	E^2xy	V^2xy	N^2xy	Y^2xy
g	g	g	b	b	b	b	g	g	g	b	b	g
g	b	b	g	b	b	b	g	b	b	g	b	g
b	g	b	g	g	g	g	b	g	b	g	b	g
b	b	b	g	b	b	b	g	g	g	b	g	b

In Table 2, the two-placed term K^2xy is interpreted as the evaluation-function "*being of both* x and y together", or "*joint being* of x with y". S^2xy is interpreted as the "*separation, divorcement* between x and y". The term X^2xy—the evaluation-function "y*'s being without* x", or "joint being of y with nonbeing of x". T^2xy—"*termination* of *x* by *y*". Z^2xy—"y's *contradiction* to (with) x". P^2xy—"*preservation, conservation, protection* of x by y". C^2xy is interpreted as the evaluation-function "*y's existence, presence in x*". E^2xy—"*equivalence, identity (of values)* of x and y". V^2xy—"*choosing and realizing* such and *only such* an element of the set {*x, y*} that is: (1) *the best* one, if both *x* and *y* are good; (2) the least bad one, if both *x* and *y* are bad; (3) the good one, if x and y have opposite values". (Thus, V^2xy means an *excluding choice and realization of only the optimal* between x and y). The term N^2xy is interpreted as the evaluation-function "*realizing neither* x *nor* y". Y^2xy is interpreted as the evaluation-function "realizing a *not-excluding-choice* result, i.e., (1) *realizing K^2xy if both x and y are good*, and (2) *realizing V^2xy otherwise*". Additional exemplifications of *two-placed evaluation-function*s may be found in [11,27,33].

To prevent possible misunderstandings of the present article, it is relevant to highlight here that in a standard interpretation of Σ + C, the signs $B_1{}^1x$, $N_1{}^1x$, $C_1{}^1x$, K^2xy, C^2xy, E^2xy, and V^2xy do not represent predicates, but *n*-placed evaluation-functions.

If an interpretation of Σ + C is defined, then expressions of the object-language of Σ + C, with the forms $(t_i = + = t_k)$, $(t_i = + = g)$, and $(t_i = + = b)$, are representations of *predicates* in Σ + C. Following the semantics of Σ + C, if t_i is a term of Σ + C, then, in a definite interpretation, a formula of Σ + C with the form $[t_i]$ is *either a true or false proposition*, "t_i exists". Thus, in a standard interpretation, formula $[t_i]$ is true if and only if t_i has the *ontological value* "e (exists)" in that interpretation. Additionally, by definition, the formula $[t_i]$ is false in a standard interpretation of Σ + C if and only if t_i has the ontological value "n (not-exists)" in that interpretation.

Following the semantics of Σ + C, in a standard interpretation of Σ + C, the formula scheme $(t_i = + = t_k)$ is a proposition of the form "t_i is *formal-axiologically equivalent* to t_k"; this proposition is true if and only if (in that interpretation) the terms t_i and t_k obtain identical *axiological values* (from the set {good, bad}) under any possible combination of *axiological values* of their *axiological variables*.

Following the semantics of Σ + C, in a standard interpretation, the formula scheme $(t_i = + = b)$ is a proposition with the form "t_i is a *formal-axiological contradiction*" (or "t_i is *formal-axiologically, or invariantly, or absolutely bad*"); this proposition is true if and only if (in that interpretation) the term t_i acquires the axiological value "bad" under any possible combination of axiological values of the axiological variables.

Following the semantics of Σ + C, in a standard interpretation, the formula scheme $(t_i = + = g)$ is a proposition with the form "t_i is a *formal-axiological law*" (or "t_i is *formal-axiologically, or invariantly, or absolutely good*"); this proposition is true if and only if (in the interpretation) the term t_i acquires the *axiological value* "good" under any possible combination of axiological values of the axiological variables.

In light of the above-given sematic meaning of $(t_i = + = t_k)$ in Σ + C, we must highlight the homonymy of the words "is", "means", "implies", "entails", and "equivalence" in natural language. On the one hand, in natural language, these words may possess their well-known formal logical meanings. On the other hand, in natural language, the same words may stand for the above-defined *formal-axiological equivalence* relation "= + =". This ambiguity of natural language must be taken into account, and these different meanings must be separated systematically; otherwise, the homonymy can lead to logic-linguistic paradoxes.

Given the above-presented formal-axiological and ontological semantics of Σ + C, it is clear that the two-valued algebraic system of formal axiology contains nothing but an abstract theory of evaluation relativity. In the theory of relativity, the *formal-axiological laws* (consistently good evaluation-functions) are *invariants* in relation to all possible transformations of interpreter V.

Thus, although it is an indisputable fact that relativity (and mutability) in empirical valuations does exist, valuation invariants (immutable universal laws of valuation relativity) do exist [34].

3.3. A Deductive Proof of the Logic Consistency of Formal Theory Σ + C

To create the proof, first of all, we must replace the meta-language of Σ + C with its object-language. To do this, we must switch from the above-formulated axiom schemes AX1–AX11 and the definition scheme DF-1 to the following axioms AX1 *–AX11 * and the definition DF-1 *, respectively.

Axiom AX-1 *: $Ap \supset (\Box q \supset q)$.
Axiom AX-2 *: $Ap \supset (\Box(p \supset q) \supset (\Box p \supset \Box q))$.
Axiom AX-3 *: $Ap \leftrightarrow (Kp \,\&\, (\neg\Diamond\neg p \,\&\, \neg\Diamond Sp \,\&\, \Box(q \leftrightarrow \Box q)))$.
Axiom AX-4 *: $Ep \leftrightarrow (Kp \,\&\, (\Diamond\neg p \vee \Diamond Sp \vee \neg\Box(q \leftrightarrow \Box q)))$.
Axiom AX-5 *: $\Box p \supset \Diamond p$.
Axiom AX-6 *: $(\Box q \,\&\, \Box\Box q) \supset q$.
Axiom AX-7 *: $\left(B_1{}^1x = + = C_1{}^1x\right) \leftrightarrow \left(G\left[B_1{}^1x\right] \leftrightarrow G\left[C_1{}^1x\right]\right)$.
Axiom AX-8 *: $\left(B_1{}^1x = + = g\right) \supset \Box G\left[B_1{}^1x\right]$.
Axiom AX-9 *: $\left(B_1{}^1x = + = b\right) \supset \Box W\left[B_1{}^1x\right]$.
Axiom AX-10 *: $(Gp \supset \neg Wp)$.
Axiom AX-11 *: $(Wp \supset \neg Gp)$.

Definition DF-1 *: $\Diamond p$ is a *name* of/for $\neg\Box\neg p$, i.e., $(\Diamond p \leftrightarrow \neg\Box\neg p)$.

Next, we give a precise definition of/for the *function* £, which is an *interpretation* of the formal theory Σ + C (it is worth emphasizing that "t" here stands for "true" and "f" stands for "false"). The interpretation function £ is defined by the following items:

(1) $£(\omega \oplus \pi) = (£\omega \oplus £\pi)$ for any formulae ω and π, and also for any binary connective $\oplus$ of classical logic;
(2) $£\neg\omega = \neg£\omega$ for any formula ω;
(3) $£Ap = f$;
(4) $£\Box q = f$;
(5) $£q = t$;
(6) $£p = t$;
(7) $£\Box(p \supset q) = f$;
(8) $£\Box p = f$;
(9) $£Kp = f$;
(10) $£\Diamond\neg p = t$;
(11) $£\Diamond Sp = t$;
(12) $£\Box(q \leftrightarrow \Box q) = f$;
(13) $£Ep = f$;
(14) $£\Diamond p = t$;
(15) $£\Box\Box q = f$;
(16) $£\left(B_1{}^1x = + = C_1{}^1x\right) = t$;
(17) $£G\left[B_1{}^1x\right] = t$;
(18) $£G\left[C_1{}^1x\right] = t$;
(19) $£\left(B_1{}^1x = + = g\right) = f$;
(20) $£\left(B_1{}^1x = + = b\right) = f$;
(21) $£\Box G\left[B_1{}^1x\right] = f$;
(22) $£\Box W\left[B_1{}^1x\right] = f$;
(23) $£Gp = t$;

(24) $\pounds Wp = f$;
(25) $\Box\neg p = f$.

Following the *interpretation* £ of the formal theory Σ + C, all the axioms AX1 *–AX11 * are true, the equivalence DF-1 * is true, and the derivation rules conserve truthfulness; consequently, a model of/for Σ + C exists, and the formal theory Σ + C is thus logically consistent.

4. Discussion

4.1. Some Special Qualities of the Logically Formalized Axiomatic Epistemology System Σ + C

Certainly, Σ + C is not a normal modal logic system following Kripke's definition of the term [35–37]. This is because the formula schemes $(\Box\beta \supset \beta)$ and $(\Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta))$ are not provable in Σ + C, and neither is Gödel's necessitation rule. However, it is possible to easily obtain a *normal* modal logic system via *modus ponens* by adding the assumption Aα to Σ + C. Nevertheless, according to Σ + C, a discussion of *empirical* knowledge and of *knowledge in general* is beyond the domain of *normal* modal logic.

Generally speaking, the formula scheme $(\alpha \leftrightarrow T\alpha)$ is not provable in Σ + C. However, adding the assumption Aα to Σ + C constructs a system (let us call it "Σ + C + Aα") in which $(\alpha \leftrightarrow T\alpha)$ is provable.

In the context of *epistemic* modal logic, it is notable that the well-known formula $(Kq \supset q)$ is not provable in Σ + C. Being accepted as a *strictly universal* principle, $(Kq \supset q)$ contradicts *evolutionary* epistemology, wherein belief revision and *knowledge revision* are addressed. Thus, Σ + C is a more realistic model of reasoning for *empirical* knowledge and *knowledge in general* (in comparison with the epistemologies accepting $(Kq \supset q)$ as a *necessarily universal* principle). Relativity, flexibility, and flow in *empirical* knowledge do exist. This is a fact of the history of cognition. Consequently, if one accepts that *evolutionary* epistemology is rational, then accepting the formula $(Kq \supset q)$ as a theorem of epistemic logic is not rational. *Evolutionary* epistemology necessitates that, in general, the modal logic of knowledge is not normal [27]. According to Σ + C, the *normal* modal logic of knowledge is rational only in the very rare cases in which the assumption Aα is true. Although $(Kq \supset q)$ is not a theorem in Σ + C, the weaker formula $(Kq \supset \Diamond q)$ is derivable in Σ + C. One significant difference between Σ and Σ + C is that the formulae $(Oq \supset \Diamond q)$, $(Gq \supset \Diamond q)$, $(Tq \supset \Diamond q)$ and $(Pq \supset \Diamond q)$, which are philosophically important, are formally derivable in Σ + C (from the axiom scheme AX-5 via relevant substitutions), but not in Σ. One of the novel benefits of the system Σ + C constructed here is its justification of the nontrivial bimodal "Kant principle", modeled by the theorem $(Op \supset \Diamond p)$. (Here I shall abstain from discussing the problem of Kant's authorship of this bimodal principle of ethics and jurisprudence, as it would be a deviation from the main theme and goal of the article). Another important benefit (and significant novelty) of Σ + C is addressed in the following section. This significant benefit (and novelty) is the main concern of this paper.

4.2. Logical Consequences of Applying Σ + C to the Conjunction of Gödel's Incompleteness Theorems and Philosophical Views of Kant and Hilbert for Proper Mathematical Knowledge System as a Whole

Now let us apply the axiomatic system Σ + C to the set of statements ST1–ST5 formulated in the introduction, which model Kant's and Hilbert's philosophies of mathematics. Within Σ + C, by means of its artificial language, the statements ST1–ST5 (formulated in the Introduction) are represented as the following formulae ST1 *–ST5 *, respectively. In these formulae, ω means either a *proper mathematical statement* or a *proper mathematical theory*, and A, T, P, and C, respectively, represent the modalities "it is *A-priori known* that . . . "; "it is *True* that . . . "; "it is *Provable* in the consistent theory that . . . "; and "it is *Consistent* that . . . ".

ST1 *: $A\omega$.
ST2 *: $(T\omega \leftrightarrow P\omega)$.
ST3 *: $(C\omega \leftrightarrow PC\omega)$.
ST4 *: $(T\omega \leftrightarrow C\omega)$.

ST5 *: $(T\omega \leftrightarrow \omega)$.

Now let us construct formal proofs of/for the following theorems of Σ + C.

Theorem 1. $(A\omega \supset (T\omega \leftrightarrow \omega))$; $(A\omega \supset (T\omega \leftrightarrow P\omega))$; $(A\omega \supset (T\omega \leftrightarrow C\omega))$; $(A\omega \supset (T\omega \leftrightarrow D\omega))$; $(A\omega \supset (P\omega \leftrightarrow D\omega))$. *The following chain of formula schemes is the formal proof of Theorem 1.*

1. $A\alpha \leftrightarrow (K\alpha \,\&\, (\neg\Diamond\neg\alpha \,\&\, \neg\Diamond S\alpha \,\&\, \Box(\beta \leftrightarrow \Omega\beta))$: axiom-scheme AX-3.
2. $A\alpha \supset (K\alpha \,\&\, (\neg\Diamond\neg\alpha \,\&\, \neg\Diamond S\alpha \,\&\, \Box(\beta \leftrightarrow \Omega\beta))$: from 1 by elimination of $\leftrightarrow$.
3. $A\alpha$: assumption.
4. $(K\alpha \,\&\, (\neg\Diamond\neg\alpha \,\&\, \neg\Diamond S\alpha \,\&\, \Box(\beta \leftrightarrow \Omega\beta)$: from 2 and 3 by *modus ponens*.
5. $\Box(\beta \leftrightarrow \Omega\beta)$: from 4 by elimination of &.
6. $(\beta \leftrightarrow \Omega\beta)$: from 3 and 5 by the rule (It is formulated as follows: $A\alpha$, $\Box\beta \mid\!- \beta$) of elimination of $\Box$.
7. $A\alpha \mid\!- (\beta \leftrightarrow \Omega\beta)$: by 1–6.
8. $A\alpha \mid\!- (\beta \leftrightarrow \Xi\beta)$: from 7 by substituting Ξ for Ω.
9. $A\alpha \mid\!- (\Xi\beta \leftrightarrow \beta)$: from 8 by commutativity of $\leftrightarrow$.
10. $A\alpha \mid\!- (\Xi\beta \leftrightarrow \Omega\beta)$: from 9 and 7 by transitivity of $\leftrightarrow$.
11. $A\alpha \mid\!- (\Omega\beta \leftrightarrow \beta)$: from 7 by commutativity of $\leftrightarrow$.
12. $\mid\!- (A\alpha \supset (\Omega\beta \leftrightarrow \beta))$: from 11 by introduction of $\supset$.
13. $\mid\!- (A\omega \supset (T\omega \leftrightarrow \omega))$: from 12 by substituting (T for Ω) and (ω for α and β).
14. $\mid\!- (A\alpha \supset (\Xi\beta \leftrightarrow \Omega\beta))$: from 10 by introduction of $\supset$.
15. $(A\omega \supset (T\omega \leftrightarrow P\omega))$: from 14 by substituting: (T for Ξ); (P for Ω); (ω for α and β).
16. $(A\omega \supset (T\omega \leftrightarrow C\omega))$: from 14 by substituting: (T for Ξ); (C for Ω); (ω for α and β).
17. $(A\omega \supset (T\omega \leftrightarrow D\omega))$: from 14 by substituting: (T for Ξ); (D for Ω); (ω for α and β).
18. $(A\omega \supset (P\omega \leftrightarrow D\omega))$: from 14 by substituting: (P for Ξ); (D for Ω); (ω for α and β).

The succession 1–6 is a formal derivation of $(\beta \leftrightarrow \Omega\beta)$ from the assumption $A\alpha$. The succession 1–10 is a formal derivation of $(\Xi\beta \leftrightarrow \Omega\beta)$ from the assumption $A\alpha$. The chain 1–13 is a formal inference of $(A\omega \supset (T\omega \leftrightarrow \omega))$. The chain 1–15 is a formal inference of $(A\omega \supset (T\omega \leftrightarrow P\omega))$. The chain 1—16 is a formal derivation of $(A\omega \supset (T\omega \leftrightarrow C\omega))$. The succession 1–17 is a formal inference of $(A\omega \supset (T\omega \leftrightarrow D\omega))$. The chain 1–18 is a formal inference of $(A\omega \supset (P\omega \leftrightarrow D\omega))$.

Corollary: from the conjunction of the above-proven Theorem 1 and Gödel's *first* theorem of incompleteness, it follows logically that $\neg A\omega$.

Theorem 2. $(A\omega \supset (C\omega \leftrightarrow PC\omega)$. *The formal proof of this is given by the following succession of formulae schemes.*

(1) $A\alpha \mid\!- (\beta \leftrightarrow \Omega\beta)$, proven by succession 1–6 in the above-constructed formal proof of Theorem 1.
(2) $A\alpha \mid\!- (\Xi\beta \leftrightarrow \Omega\Xi\beta)$, inferred from (1) by substituting ($\Xi\beta$ for β).
(3) $\mid\!- (A\alpha \supset (\beta \leftrightarrow \Omega\beta)$, inferred from (2) by the rule of introduction of $\supset$.
(4) $\mid\!- (A\omega \supset (C\omega \leftrightarrow PC\omega)$, inferred from (3) by substituting (C for Ξ); (P for Ω) and (ω for α and β).

Corollary: from the conjunction of the above-proven Theorem 2 and Gödel's *second* theorem of incompleteness, it follows logically that $\neg A\omega$.

Professional mathematicians and logicians are often skeptical of Leibniz's belief (dream) that developing a modal *propositional* logic can help in discussing, precisely formulating, and effectively solving some (if not many) proper philosophical questions. Why this concrete, propositional logic? Why not, for instance, *first-order predicate* logic (or some more general, richer, or stronger logic system)? These questions are natural and nontrivial. My answer is the following. According to the above-proven Theorem 1, within the formal axiomatic epistemology system Sigma + C, it is formally provable that

(Aω ⊃ (Tω ↔ Dω)), (Aω ⊃ (Pω ↔ Dω)). In the standard epistemological interpretation, the theorem schemes (Aω ⊃ (Tω ↔ Dω)), (Aω ⊃ (Pω ↔ Dω)) mean that an *a priori* knowledge system is *decidable*. As *first-order predicate* logic is notoriously not decidable, it cannot be a proper logical part (logical subsystem) of a formalized system of proper *a priori* knowledge. Consequently, *first-order predicate* logic cannot offer a proper logical basis of *universal* epistemology that is *common to both* the empirical and *a priori* knowledge subsystems. Many other respected and consistent logic systems (that are more general, rich, and powerful than *first-order predicate* logic) are either not complete, or are *not decidable*; consequently, they are not able to provide a proper logical basis of/for a logically formalized *universal* epistemology that *consistently combines* both empiricism and a priority in the philosophy of knowledge. It is therefore quite reasonable to use propositional logic for the role in question, because propositional logic is consistent, complete, and *decidable,* and consequently, it is *compatible* with the *a priority* of epistemology (while *first-order predicate* logic and the *a priority* of epistemology are not compatible, given the theorem schemes (Aω ⊃ (Tω ↔ Dω))), (Aω ⊃ (Pω ↔ Dω)). However, I agree that *first-order predicate* logic is an adequate—and in some cases may even be the best—proper logical basis for the formalization of many *empirical* knowledge systems.

5. Conclusions

Kant emphasized that proper mathematical knowledge is purely *a priori*. This is especially highlighted not only in his *Critique of Pure Reason* [1], but also in his *Prolegomena* [38]. At first glance, Kantean philosophy of mathematics looks quite reasonable and veracious; however, from the conjunction of the axiomatic epistemology system Σ + C and the famous incompleteness theorems of Gödel, it follows that Kant's statement of the *a priority* of mathematical knowledge is false. Moreover, it is formally provable in Σ + C that $(\mathrm{A}\omega \supset (\omega \leftrightarrow \mathrm{T}\omega))$; consequently, those who do not agree with the equivalence $(\omega \leftrightarrow \mathrm{T}\omega)$ must also disagree with Kant's assertion (modeled by Aω). Additionally, it is formally provable in Σ + C that $(\mathrm{A}\omega \supset (\mathrm{T}\omega \leftrightarrow \mathrm{C}\omega))$; consequently, if one negates $(\mathrm{T}\omega \leftrightarrow \mathrm{C}\omega)$, then one must also negate Aω. This means that mathematical knowledge as a whole is not an *a priori* system—rather, it is an *empirical* one, and only some small (but nevertheless very important) aspects are *a priori*. As such, it is evident and formally demonstrable that, in Σ + C, Hilbert's ideal and program of philosophical grounding mathematics as a *self-sufficient* system *logically follow* from Kant's presumption that any proper mathematical statements and systems of knowledge are *a priori*. If Kant's presumption was correct, then, by means of Σ + C, Hilbert's ideal and program would be well grounded, convincingly explained, and totally vindicated. However, Kant's presumption is wrong; hence, the significance of Hilbert's ideal and program is limited. However, notwithstanding its limitations, it is fairly adequate, and works effectively within its own reduced domain of applicability.

In principle, it seems that the conclusions concerning Kantian *a priority*, obtained via = the complex formal axiomatic theory Σ + C, may be obtained more easily via natural logic. However, in my opinion, this is only true in the case of some relations. Sometimes, simplicity is too expensive, coming at the cost of significant losses of precision and rigor. Unfortunately, in the humanities, confined to natural language and natural logic, Kant's *a priority* is barely addressed, and any conclusions on it are not convincing. The present article does not mainly concern Kant as a representative of the humanities, but rather concerns D. Hilbert as a formally minded mathematician striving to establish mathematics as a self-sufficing system, who exploits Kant's *a priority* to do so. In relation to Hilbert's program, an analogous skepticism can be developed. There are many respectable creative mathematicians (for example, H. Poincaré [39]) who use intuition, construction, and natural logic exclusively; they have a critical attitude towards D. Hilbert's formalism and B. Russell's logicism in addressing the philosophical foundations of mathematics and physics. Nevertheless, in my opinion, Hilbert's formalism is heuristically important—sometimes, it leads to ground-breaking nontrivial scientific discoveries; for instance, in meta-mathematics and mathematical physics.

As such, the above-presented (but not exhaustively addressed) trend of constructing and investigating multimodal, formal, axiomatic philosophical systems from Ξ, Σ, and Σ + C can also promise nontrivial discoveries in philosophical knowledge, for example, in proper philosophical formal ontology, formal epistemology, and formal axiology. Such discoveries may have applications in the philosophies of science, logic, ethics, aesthetics, and jurisprudence, and in philosophical theology. They may also be applicable to physics. At the very end of the 19th century and in the first half of the 20th century, many celebrated mathematicians were interested in addressing the difficult (even paradoxical) situation of mathematical physics. D. Hilbert was among these famous mathematicians. He tried to apply his idea of logically formalized axiomatic theories to the development of mathematical physics by creating a solid basis for it [20,40]. I have attempted to continue this by applying the formal axiomatic theory Σ to a philosophical grounding of classical mechanics [28,29]. The results are as follows. I have constructed "a formal deductive inference of the law of inertia in Σ from the assumption of knowledge a-priori-ness" [28]. I have also constructed "a formal deductive inference of the law of conservation of energy from assuming a-priori-ness of knowledge in Σ" [29]. However, interesting themes for discussion remain, and difficult theoretical problems await solutions. I propose that the research submitted here may be advanced in two ways.

Firstly, in the future, it would be worth attempting to further develop the formal axiomatic epistemology and axiology theory Σ + C by modifying it. However, it is not quite clear whether the above-presented set of axiom schemes of Σ + C is sufficient for adequately mathematically modeling the possible application domains. At present, it is still not possible to exactly formulate some aspects of the problem of completeness of Σ + C. The *syntax* of Σ + C has been sufficiently elaborated and represented, as have the *formal* aspects of the semantics of Σ + C. However, the *content* of the semantics of Σ + C, namely, the relationship between the standard interpretation of Σ + C and the as-yet indefinite (unrestricted) application domain of Σ + C, needs to be investigated further. Some of the content-related intuitions underlying the formal theory probably remain unformulated because they are too vague and not well recognized. I feel that the *content* aspect of the semantics of Σ + C (its relationship to the external world) has yet to undergo significant development and elaboration. This analysis of the incompletely defined subject matter of the axiomatic epistemology and axiology theory Σ + C may transform Σ + C into a qualitatively new formal theory. In particular, in the future, some qualitatively new nontrivial axiom schemes may be added to Σ + C.

Secondly, in the future, the possibility of applying the two-valued algebra of formal axiology to some other necessarily universal laws of nature should be addressed; for example, to the laws of conservation in physics. Such hypothetical applications would be theoretically interesting, although we should be prepared for the failure of some of these intellectual experiments. Let us try and see. Regardless, I believe that the application of Σ + C and its modifications to philosophically grounding physics is worth discussing and developing further. Further hypothetical modifications of Σ + C and the resulting applications are future vistas.

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